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Nonlinear Damped Oscillators on Riemannian Manifolds: Fundamentals*

FIORI Simone

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Abstract The classical theory of mass-spring-damper-type dynamical systems on the ordinary flat space $\mathbb{R}^3$ may be generalized to higher-dimensional Riemannian manifolds by reformulating the basic underlying physical principles through differential geometry. Nonlinear dynamical systems have been studied in the scientific literature because they arise naturally from the modeling of complex physical structures and because such dynamical systems constitute the basis for several modern applications such as the secure transmission of information. The flows of nonlinear dynamical systems may evolve over time in complex, non-repeating (although deterministic) patterns. The focus of the present paper is on formulating the general equations that describe the dynamics of a point-wise particle sliding on a Riemannian manifold in a coordinate-free manner. The paper shows how the equations particularize in the case of some manifolds of interest in the scientific literature, such as the Stiefel manifold and the manifold of symmetric positive-definite matrices.

Keywords Nonlinear (active/passive) damping, nonlinear oscillator, Riemannian manifold.

1 Introduction

Nonlinear dynamical systems have been widely studied in the scientific literature because they arise naturally from the modeling of complex physical structures (such as in the study of vacuum tubes[1]) and because such dynamical systems constitute the basis for several modern applications, such as in the secure transmission of information[2]. The state of nonlinear dynamical systems may evolve over time in complex, non-repeating (albeit deterministic) patterns. Contributions from the scientific literature suggest, with an increasing degree of complexity and generality, how the simplest single-oscillator models may be generalized to complex single- or coupled-oscillators models within application-oriented and purely-theoretical research scopes[3–6].

FIORI Simone
*Dipartimento di Ingegneria dell'Informazione, Facoltà di Ingegneria, Università Politecnica delle Marche, Via Brecce Bianche, Ancona I-*60131, *Italy.* Email: s.fiori@univpm.it.
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Most nonlinear oscillator models appear in the scientific literature as dynamical systems involving a single real variable. The simplest model is perhaps the linear harmonic oscillator described by the dynamical system:

$$\ddot{x} + \Omega_0^2 x = 0, \tag{1}$$

where $x \in \mathbb{R}$ is the fundamental variable and $\Omega_0 > 0$ denotes the natural oscillation frequency of the harmonic oscillator. The state $x = x(t)$ is a function of time, with $t \in \mathbb{R}$, and the overdots denote derivatives with respect to the temporal variable t. As no damping is present, the harmonic oscillator conserves its initial energy indefinitely. The same consideration applies to the simple pendulum equation $\ddot{x} + \Omega_0^2 \sin x = 0$, that may be rewritten as the dynamical system:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\dfrac{dV_x}{dx}, \end{cases} \tag{2}$$

with potential energy function $V : \mathbb{R} \to \mathbb{R}$ defined as $V_x \stackrel{\text{def}}{=} \Omega_0^2(1 - \cos x)$.

A well-studied nonlinear damped dynamical system that arose from the analysis of vacuum tubes is the van der Pol oscillator, described by the second-order differential equation:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \Omega_0^2 x = 0, \tag{3}$$

where again $x \in \mathbb{R}$ and $\mu > 0$ denotes a parameter that indicates the nonlinearity and strength of the damping. When $\mu = 0$, the equation (3) collapses into the equation of a harmonic oscillator (1). When $\mu > 0$, the system will eventually enter a limit cycle. Near the origin $(x, \dot{x}) = (0, 0)$ the system is unstable while far from the origin the system is damped. The equation (3) may be rewritten as the dynamical system:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\dfrac{dV_x}{dx} - \mu(x^2 - 1)v, \end{cases} \tag{4}$$

where the potential $V : \mathbb{R} \to \mathbb{R}$ is defined as $V_x \stackrel{\text{def}}{=} \frac{1}{2}\Omega_0^2 x^2$ and $\Omega_0 > 0$. Figure 1 illustrates the behavior of a van der Pol oscillator: After an initial transient, the system enters a stable cycle in the phase plane corresponding to a periodic oscillation; the total energy changes over time, accordingly.

The Van der Pol oscillator model arises as a special case of the FitzHugh-Nagumo model[7] which, in turn, is a simplified version of the Hodgkin-Huxley model that explains in a detailed manner activation and deactivation dynamics of spiking neurons. A related example arises from the examination of the driven van der Pol oscillator, described by the differential system:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\dfrac{dV_x}{dx} - \mu(x^2 - 1)v + A\sin(\Omega t), \end{cases} \tag{5}$$

where $A > 0$ and $\Omega > 0$. In the present case, the non-linear damping term comprises a quantity that depends sinusoidally from the temporal variable t. The particular form of the damping

causes a decrease of the amplitude of the great oscillations and an increase of the amplitude of the small oscillations. The original Van der Pol oscillator was generalized in different ways, as in the paper [8] that considers the introduction of fractional derivatives in the dynamical system.

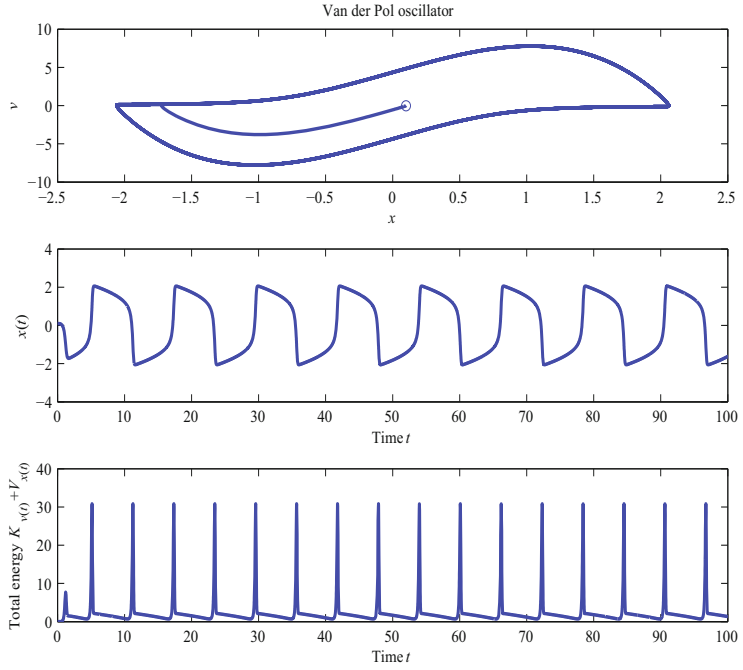

Figure 1 Behaviour of the Van der Pol oscillator (4). The values of the parameters are $\mu = 5$ and $\Omega_0 = \frac{\pi}{5}$. Top panel: Trajectory in the phase-space (the open circle denotes the starting point $(x(0), \dot{x}(0)) = \left(\frac{1}{10}, 0\right)$). Middle panel: Values of the variable x versus time. Bottom panel: Total energy $K_{v(t)} + V_{x(t)}$, where the term $K_{v(t)} \stackrel{\text{def}}{=} \frac{1}{2}v^2(t)$

Another well-studied nonlinear system that exhibits a complex behavior is the Duffing oscillator. Duffing's dynamical system reads:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\dfrac{dV_x}{dx} - \mu v + A\sin(\Omega t), \end{cases} \tag{6}$$

where $A > 0$, $\Omega > 0$ and the potential energy function $V : \mathbb{R} \to \mathbb{R}$ is defined according to one of the three known models, namely:

- Hard Duffing oscillator: The potential is defined as $V_x \stackrel{\text{def}}{=} \frac{1}{2}\Omega_0^2 x^2 + \frac{1}{4}\alpha\Omega_0^2 x^4$, with $\Omega_0 > 0$ and $\alpha > 0$.
- Double-well Duffing oscillator: The potential is defined as $V_x \stackrel{\text{def}}{=} -\frac{1}{2}\Omega_0^2 x^2 + \frac{1}{4}\alpha\Omega_0^2 x^4$, with $\Omega_0 > 0$ and $\alpha > 0$.

- Soft Duffing oscillator: The potential is defined as $V_x \stackrel{\text{def}}{=} \frac{1}{2}\Omega_0^2 x^2 - \frac{1}{4}\alpha\Omega_0^2 x^4$, with $\Omega_0 > 0$ and $\alpha > 0$.

The dynamical system (6) describes the motion of a damped oscillator with a more complicated potential than in the simple harmonic oscillator. It models, for example, a spring pendulum whose spring's stiffness does not exactly obey Hooke's law. A comparison between the Duffing oscillator (6) and the driven Van der Pol system (5) reveals that the linear damping term $-\mu v$ in the Duffing system is replaced by the non-linear damping term $-\mu(x^2-1)v$ in the Van der Pol system. The contribution [9] describes a generalized double-well Duffing oscillator endowed with non-linear damping, namely, the system:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\Omega_0^2 x + \alpha\Omega_0^2 x^3 - \mu v|v|^{\eta-1} + A\sin(\Omega t), \end{cases} \tag{7}$$

where $\eta \geq 1$ is termed damping exponent. The case $\eta = 1$ corresponds to the linear damping case in the system (6). A similar analysis was performed in [10] about the universal escape oscillator endowed with the same nonlinear damping term. Figure 2 illustrates the behavior of a double-well Duffing oscillator with non-linear damping.

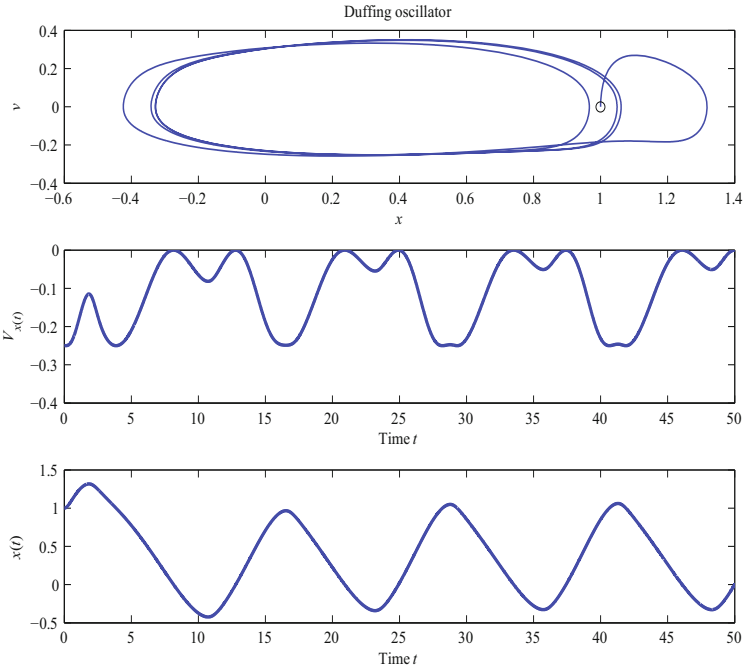

Figure 2 Behaviour of a double-well Duffing oscillator (7) with non-linear damping. The values of the parameters are: $\mu = 10$, $A = 1$, $\Omega = 0.5$, $\eta = 2$, $\Omega_0 = 1$, $\alpha = 1$. Top panel: Trajectory in the phase-space (the open circle denotes the starting point $(x(0), \dot{x}(0)) = (1, 0)$). Middle panel: Values of the potential energy function over the trajectory. Bottom panel: Values of the variable x

Examples of dynamical systems involving more than one variable are known in the scientific literature. The best known example is perhaps the Lorenz oscillator[11]. The Lorenz oscillator is a nonlinear, three-dimensional dynamical system that generates a complex flow, easily recognizable from its lemniscate-like shape. A Lorenz oscillator is described by the dynamical system:

$$\begin{cases} \dot{x}_1 = \sigma(x_2 - x_1), \\ \dot{x}_2 = x_1(\rho - x_3) - x_2, \\ \dot{x}_3 = x_1 x_2 - \beta x_3, \end{cases} \tag{8}$$

where $\sigma, \rho, \beta > 0$ and $(x_1, x_2, x_3) \in \mathbb{R}^3$. For certain values of the parameters, the system exhibits a complex behavior. The Lorenz equations were derived from the simplified equations of convection rolls arising in the study of the atmosphere and has important implications to what concerns climate and weather prediction. Figure 3 illustrates the behavior of the Lorenz oscillator.

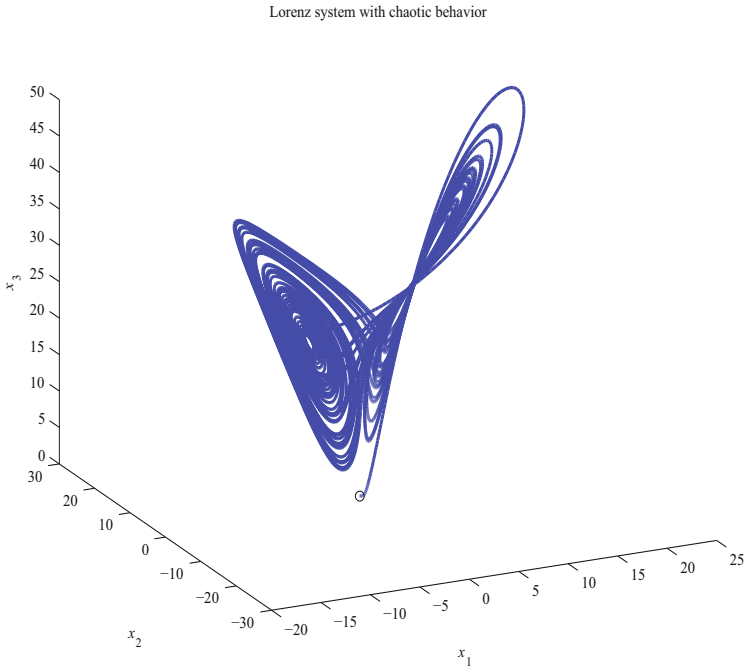

Figure 3 Behaviour of the Lorenz oscillator (8) with non-linear damping. The values of the parameters are $\rho = 28$, $\sigma = 10$, $\beta = \frac{8}{3}$. The figure shows the trajectory in the space $\mathbb{R}^3$, where the open circle denotes the starting point $(x_1, x_2, x_3) = (0, 1, 1.05)$

A further nonlinear, three-dimensional, deterministic dynamical system known from the scientific literature is the Rabinovich-Fabrikant oscillator[12]. It is described by a set of three coupled ordinary differential equations comprising two parameters, which may exhibit a complex behavior for certain values of the parameters, while for other values of the parameters its flow may tend to a stable periodic orbit. Likewise, a Rössler oscillator[13] has some similarities

with (but it is simpler than) the Lorenz oscillator. The (originally theoretical) equations were found to be useful in modeling the equilibrium in chemical reactions. In addition, a three-dimensional non-linear oscillation model well known in circuit theory is the Colpitts oscillator, built up of a bipolar junction transistor and a resonant network consisting of an inductor and two capacitors[14].

The Lorenz system (8) may be rewritten as a second-order dynamical system by deriving both sides of each equation with respect to the parameter t:

$$\begin{cases} \ddot{x}_1 = -\sigma\dot{x}_1 + \sigma\dot{x}_2, \\ \ddot{x}_2 = (\rho - x_3)\dot{x}_1 - \dot{x}_2 - x_1\dot{x}_3, \\ \ddot{x}_3 = x_2\dot{x}_1 + x_1\dot{x}_2 - \beta\dot{x}_3. \end{cases} \tag{9}$$

Appropriate initial conditions need to be set up. By defining $x \stackrel{\text{def}}{=} [x_1 \ x_2 \ x_3]^{\mathrm{T}}$, the dynamical system (9) may be cast in the compact form:

$$\begin{cases} \dot{x} = v, \\ \\ \dot{v} = -C_x v, \quad C_x \stackrel{\text{def}}{=} \begin{bmatrix} \sigma & -\sigma & 0 \\ x_3 - \rho & 1 & x_1 \\ -x_2 & -x_1 & \beta \end{bmatrix}. \end{cases} \tag{10}$$

The Lorenz system assumes the form of a set of purely non-linearly damped oscillators on a flat space endowed with a constant potential energy function. A general form for coupled scalar oscillators in $\mathbb{R}^n$ may be drawn from the study [15] and casts as:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -(J_V)_x - \varphi_x(v) + uf(t), \end{cases} \tag{11}$$

where $x \in \mathbb{R}^n$ is the vector of variables, $V : \mathbb{R}^n \to \mathbb{R}$ is a potential energy function, $(J_V)_x \in \mathbb{R}^n$ denotes the Jacobian vector of partial derivatives with respect to the coordinates x, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator in the variable v (possibly nonlinear in the variable x) that represents damping and $f : \mathbb{R} \to \mathbb{R}$ represents a forcing term in the fixed direction $u \in \mathbb{R}^n$.

The aim of the present manuscript is to explore the possibility of extending the theory of nonlinear oscillators to Riemannian manifolds on the basis of the previously-recalled oscillator models. Much of the discussion in the physics and engineering literature concerning damped systems focuses on systems subjected to viscous damping even though viscous damping occurs rarely in real physical systems. Other types of dissipative forces exist in real systems and will replace the linear damping term, in order to explore richer damping phenomena. The present manuscript focuses on the theoretical grounds that lead to an extension of the theory of nonlinear damped oscillators on low-dimensional Euclidean spaces to high-dimensional curved (Riemannian) manifolds. In particular, the present manuscript focuses on the derivation of a general framework to develop a theory of nonlinear dynamical systems on Riemannian manifolds

by a stationary-action principle. The obtained dynamical system will be expressed in terms of a second-order differential equation on manifold.

The present paper is of theoretical nature and hence does not illustrate any specific application-oriented implementation of the developed theory.

The paper is organized as follows. Section 2 recalls differential-geometrical instruments and expresses an extended Hamiltonian system which includes several nonlinear damping phenomena. Section 3 illustrates the studied dynamical system theory by showing the explicit structure of the general systems for specific manifolds of interest in the scientific literature. Section 4 concludes the paper.

2 Non-Linear Oscillators on Manifolds

Subsection 2.1 recalls notions of differential geometry that are instrumental in the writing of the equations of the non-linear oscillators on manifolds. Subsection 2.2 presents a general theory for deriving dynamical systems on manifolds and introduces a non-linear oscillation system on manifold. Subsection 2.3 discusses some choices of potential functions and presents the related calculations.

2.1 Notions and Notation of Differential Geometry

For the theory of differentiable manifolds, readers may consult the series of books[16].

Let M denote a real differentiable manifold of dimension r. In local coordinates, a point $x \in M$ is denoted by $(x^1, x^2, \cdots, x^r)$. At a point $x \in M$, the tangent space to the manifold M is denoted as T_xM and represents the vector space of dimension r spanned by all tangent vectors to all smooth curves on M passing through the point x. The canonical basis of a tangent space T_xM is denoted by $(\partial_1, \partial_2, \cdots, \partial_r)$ where $\partial_\sigma = \partial_\sigma(x)$. The symbol TM denotes the tangent bundle defined as $TM \stackrel{\text{def}}{=} \{(x, v) | x \in M, v \in T_xM\}$. The cotangent space to the manifold M at a point $x \in M$ is denoted as $T_x^\star M$ and represents the set of linear functions from T_xM to $\mathbb{R}$. Elements of a cotangent space are termed cotangent vectors. The canonical basis of the cotangent space $T_x^\star M$ is denoted by $(dx^1, dx^2, \cdots, dx^r)$. In the present Section, the Einstein summation convention is in force: In an expression where repeated indexes occur, summation over those indexes is implied. Likewise, the standard notation to distinguish covariant/contravariant indexes is made use of.

A Riemannian manifold M is endowed with a bilinear, positive-definite form $\mathbb{G}_x : T_xM \times T_xM \to \mathbb{R}$. In local coordinates, the bilinear form $\mathbb{G}$ is expressed by the components of the metric tensor $\mathbb{G}_{\sigma\tau} = \mathbb{G}_{\sigma\tau}(x) \stackrel{\text{def}}{=} \mathbb{G}_x(\partial_\sigma, \partial_\tau)$ termed local metric. Therefore, expressing two tangent vectors as $w = w^\sigma \partial_\sigma$ and $v = v^\sigma \partial_\sigma$, bilinearity implies $\mathbb{G}_x(w, v) = \mathbb{G}_{\sigma\tau}(x) w^\sigma v^\tau$. The components of the inverse metric tensor are denoted by $\mathbb{G}^{\sigma\tau}$. A local metric $\mathbb{G}_x$ also defines a local norm $\|v\|_x \stackrel{\text{def}}{=} \sqrt{\mathbb{G}_x(v, v)}$, for $v \in T_xM$. The metric 'flat' operator $\mathbb{G}^\flat$ converts a tangent vector into a cotangent vector, namely, $\mathbb{G}_x^\flat : T_xM \to T_x^\star M$. In local coordinates, let $v = v^\sigma \partial_\sigma$, then $\mathbb{G}_x^\flat(v) = \mathbb{G}_{\sigma\tau} v^\sigma dx^\tau$. Its inverse is termed 'sharp' operator and is denoted by $\mathbb{G}^\sharp$. In local coordinates, let $u = u_\sigma dx^\sigma$, then $\mathbb{G}_x^\sharp(u) = \mathbb{G}^{\sigma\tau} u_\sigma \partial_\tau$. Let $\psi : M \to \mathbb{R}$ denote a differentiable function. The differential of a function $\psi : M \to \mathbb{R}$ at a point $x \in M$ is denoted by $d_x\psi \in T_x^\star M$.

The Riemannian gradient of the function ψ with respect to a metric $\mathbb{G}$, evaluated at the point $x \in M$ is $(\mathbb{G}_x^\sharp \circ d_x)\psi$. The differential, the Riemannian gradient and the Euclidean gradient of the function ψ are related by:

$$d_x\psi(v) = \frac{\partial \psi}{\partial x^\sigma} v^\sigma = \mathbb{G}_x((\mathbb{G}_x^\sharp \circ d_x)\psi, v), \tag{12}$$

where the second equality expresses the compatibility of the gradient and the metric. As a consequence of the compatibility condition, it turns out that the Riemannian gradient of a function $\psi : M \to \mathbb{R}$ at a point $x \in M$ may be computed as $\mathbb{G}_x^\sharp \left(\frac{\partial \psi}{\partial x^\sigma} dx^\sigma \right)$.

Symbol $\mathfrak{X}(M)$ denotes the set of vector fields on M. A vector field[†] $\mathfrak{F} \in \mathfrak{X}(M)$ is a map $\mathfrak{F} : x \in M \mapsto \mathfrak{F}(x) \in T_xM$. The covariant derivative (or connection) of a vector field $\mathfrak{F} \in \mathfrak{X}(M)$ in the direction of a vector $v \in T_xM$ is denoted as $\nabla_v \mathfrak{F}$. The covariant derivative is defined axiomatically by the following properties:

$$\begin{cases} \nabla_{\psi v + \varphi w} \mathfrak{F} = \psi \nabla_v \mathfrak{F} + \varphi \nabla_w \mathfrak{F}, \\ \nabla_v (\mathfrak{F} + \mathfrak{G}) = \nabla_v \mathfrak{F} + \nabla_v \mathfrak{G}, \\ \nabla_v (\psi \mathfrak{F}) = \psi \nabla_v \mathfrak{F} + d_x\psi(v) \mathfrak{F}, \end{cases} \tag{13}$$

for a point $x \in M$, vector fields $\mathfrak{F}, \mathfrak{G} \in \mathfrak{X}(M)$, tangent vectors $v, w \in T_xM$ and scalar functions $\psi, \varphi : M \to \mathbb{R}$, with ψ differentiable. The fundamental relationship for the connection is:

$$\nabla_{\partial_\sigma} \partial_\tau = \Gamma_{\sigma\tau}^\alpha \partial_\alpha, \tag{14}$$

where the quantities $\Gamma_{\sigma\tau}^\alpha : M \to \mathbb{R}$ are termed Christoffel symbols of the second kind and describe completely the structure of the connection. The covariant derivative $\nabla_{\partial_\sigma} \partial_\tau$ measures the change of the elementary vector field $\partial_\tau = \partial_\tau(x)$ in the direction $\partial_\sigma(x)$. By the axioms (13), it is readily obtained that:

$$\nabla_{\mathfrak{F}^\sigma \partial_\sigma} (\mathfrak{G}^\tau \partial_\tau) = \mathfrak{F}^\sigma \left(\Gamma_{\sigma\tau}^\alpha \mathfrak{G}^\tau + \frac{\partial \mathfrak{G}^\alpha}{\partial x^\sigma} \right) \partial_\alpha, \tag{15}$$

where the functions $\mathfrak{F}^\sigma = \mathfrak{F}^\sigma(x)$ and the functions $\mathfrak{G}^\tau = \mathfrak{G}^\tau(x)$ are the components of two vector fields in $\mathfrak{X}(M)$ in the basis $\{\partial_\sigma\}$. The covariant derivative of a vector field $\mathfrak{F} \in \mathfrak{X}(M)$ along a vector $v \in T_xM$ may be extended to the covariant derivative of a vector field $\mathfrak{F}$ along a vector field $\mathfrak{G}$ by the rule $(\nabla_{\mathfrak{G}} \mathfrak{F})(x) \stackrel{\text{def}}{=} (\nabla_{\mathfrak{G}(x)} \mathfrak{F})(x)$. Such a rule defines a connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$.

A vector field $\varphi \in \mathfrak{X}(M)$ that depends on a parameter, namely $\varphi_x : T_xM \to T_xM$, is positive-definite if $\mathbb{G}_x(\varphi_x(v), v) > 0$ for every $v \in T_xM - \{0\}$ and every $x \in M$.

The Christoffel symbols of the second kind may be specified arbitrarily and give rise to an arbitrary connection. On a Riemannian manifold with metric $\mathbb{G}$, there exists a unique covariant derivative termed Levi-Civita connection, denoted by $\overset{\mathbb{G}}{\nabla} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$.

[†]In the present section, the symbols $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{V}$ and $\mathfrak{A}$ are made use of. They denote an 'F', a 'G', a 'V' and an 'A', respectively, in Gothic typeface.

The Christoffel symbols of the second kind of the Levi-Civita connection associated with the metric tensor of components $\mathbb{G}_{\sigma\tau}$ are defined as:

$$\overset{\mathbb{G}}{\Gamma}{}^{\alpha}_{\sigma\tau} \overset{\text{def}}{=} \frac{1}{2}\mathbb{G}^{\alpha\beta}\left(\frac{\partial \mathbb{G}_{\beta\tau}}{\partial x^{\sigma}} + \frac{\partial \mathbb{G}_{\sigma\beta}}{\partial x^{\tau}} - \frac{\partial \mathbb{G}_{\sigma\tau}}{\partial x^{\beta}}\right). \tag{16}$$

The associated Christoffel form $\overset{\mathbb{G}}{\Gamma}_x$, with domain $T_xM \times T_xM$, is defined in local coordinates by $[\overset{\mathbb{G}}{\Gamma}_x(v,w)]^{\alpha} \overset{\text{def}}{=} \overset{\mathbb{G}}{\Gamma}{}^{\alpha}_{\sigma\tau} v^{\sigma} w^{\tau}$.

The notion of covariant derivative is closely tied to the notion of parallel translation (or transport)[‡] along a curve. On a Riemannian manifold M with connection ∇, fix a smooth curve $\gamma : \mathbb{I} \to M$ with $0 \in \mathbb{I} \subset \mathbb{R}$. The parallel translation operator $\mathrm{P}_{\gamma}^{s\to t} : T_{\gamma(s)}M \to T_{\gamma(t)}M$ associated with the curve γ is a linear isomorphism for every $s, t \in \mathbb{I}$. The parallel translation map depends smoothly on its arguments and is such that $\mathrm{P}_{\gamma}^{t\to t}$ is the identity map in $T_{\gamma(t)}M$ and $\mathrm{P}_{\gamma}^{u\to t} \circ \mathrm{P}_{\gamma}^{s\to u} = \mathrm{P}_{\gamma}^{s\to t}$ for every $s, u, t \in \mathbb{I}$. By definition, parallel translation is an isometry, namely, given a curve $\gamma : \mathbb{I} \to M$ with $0 \in \mathbb{I}$, $\gamma(0) = x$ and vectors $v, w \in T_xM$, it holds that $\mathbb{G}_{\gamma(t)}(P_{\gamma}^{0\to t}(v), P_{\gamma}^{0\to t}(w)) = \mathbb{G}_x(v,w)$ for every $t \in \mathbb{I}$. In fact, parallel transport preserves the angle between transported vectors.

Assign a vector $v \in T_xM$ and consider a curve $\gamma : \mathbb{I} \to M$ such that $\dot{\gamma}(0) = v$. The covariant derivative of a vector field $\mathfrak{F} \in \mathfrak{X}(M)$ in the direction v is related to the parallel translation operator by:

$$\nabla_v \mathfrak{F} = \lim_{\varepsilon \to 0} \frac{\mathrm{P}_{\gamma}^{\varepsilon\to 0}[\mathfrak{F}(\gamma(\varepsilon))] - \mathfrak{F}(\gamma(0))}{\varepsilon} = \frac{d}{dt}\mathrm{P}_{\gamma}^{t\to 0}\mathfrak{F}(\gamma(t))\Big|_{t=0}. \tag{17}$$

A smooth curve $\gamma : \mathbb{I} \to M$ has a 'geometric velocity' vector field $\mathfrak{V}(\gamma(t)) \overset{\text{def}}{=} \dot{\gamma}(t)$ naturally associated. A vector field $\mathfrak{F} \in \mathfrak{X}(M)$ is said to be parallel along a curve $\gamma : \mathbb{I} \to M$ if it satisfies the condition $\mathrm{P}_{\gamma}^{s\to t}\mathfrak{F}(\gamma(s)) = \mathfrak{F}(\gamma(t))$, for any $s, t \in \mathbb{I}$. In terms of covariant derivative, the parallelism condition reads $\nabla_{\dot{\gamma}}\mathfrak{F} = 0$. It turns out from the property (17) that the parallel translation operator, that allows parallel-translating a tangent vector $w \in T_xM$ along a curve $\gamma : \mathbb{I} \to M$ such that $\gamma(0) = x$, is $\mathrm{P}_{\gamma}^{0\to t}(w) = \mathfrak{F}(\gamma(t))$ where the vector field $\mathfrak{F}$ is the solution of the differential equation $\nabla_{\dot{\gamma}}\mathfrak{F} = 0$. Hence, a Levi-Civita connection $\overset{\mathbb{G}}{\nabla}$ defines a parallel translation operator $\overset{\mathbb{G}}{\mathrm{P}}$. The parallelism equation reads:

$$\frac{d\overset{\mathbb{G}}{\mathrm{P}}{}_{\gamma}^{0\to t}(w)}{dt} + \overset{\mathbb{G}}{\Gamma}_{\gamma}\left(\overset{\mathbb{G}}{\mathrm{P}}{}_{\gamma}^{0\to t}(w), \frac{d\gamma}{dt}\right) = 0. \tag{18}$$

The notion of geodesic curve generalizes the notion of straight line of Euclidean spaces. A distinguishing feature of a straight line of an Euclidean space is that it translates parallel to itself, namely, it is self-parallel. The notion of 'straight line' on a curved space inherits such a distinguishing feature. A geodesic on a manifold M with connection $\overset{\mathbb{G}}{\nabla}$ and associated parallel

[‡]Traditionally, the term "parallel translation" is reserved to the rigid translation of a vector on a flat space such as $\mathbb{R}^3$, while the term "parallel transport" denotes its counterpart on tangent bundles. In the present paper, such distinction is ignored and the term "translation" is used as a unifying word.

translation operator $\overset{\mathbb{G}}{\mathrm{P}}$, is a curve γ such that $\dot{\gamma}$ is parallelly translated along γ itself, namely, for every $s, t \in \mathbb{I}$, it holds that:

$$\overset{\mathbb{G}}{\mathrm{P}}{}_{\gamma}^{s \to t}(\dot{\gamma}(s)) = \dot{\gamma}(t). \tag{19}$$

By the parallelism condition expressed through the covariant derivative, it is seen that a geodesic curve is the solution of the equation $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ with appropriate initial conditions. The geodesic equation reads:

$$\frac{d^2\gamma}{dt^2} - \overset{\mathbb{G}}{\Gamma}_{\gamma}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) = 0. \tag{20}$$

A geodesic curve $\gamma : \mathbb{I} \to M$ that satisfies the initial conditions $\gamma(0) = x \in M$ and $\dot{\gamma}(0) = v \in T_xM$ is denoted as $\gamma_{x,v}(t)$. A smooth curve $\gamma : \mathbb{I} \to M$ has a 'geometric acceleration' vector field $\mathfrak{A}(\gamma(t)) \overset{\text{def}}{=} \nabla_{\dot{\gamma}}\dot{\gamma}$ naturally associated. The geometric acceleration associated with a geodesic curve is everywhere zero, hence its scalar geometric velocity $\|\mathfrak{V}(\gamma(t))\|_{\gamma(t)}$ keeps constant over time.

Given a geodesic line $\gamma_{x,v}(t)$, with $t \in [0, 1]$, a manifold exponential $\exp : TM \to M$ is defined as $\exp_x(v) \overset{\text{def}}{=} \gamma_{x,v}(1)$. It maps a tangent vector $v \in T_xM$ to a point $y = \exp_x(v)$ that belongs to a neighbor of the point $x \in M$. Its inverse 'log' is defined only locally and is termed manifold logarithm. Given points $x, y \in M$, it computes a tangent vector $v = \log_x(y) \in T_xM$ such that $\exp_x(v) = y$. A manifold exponential map depends on the chosen metric through the dependency chain:

$$\text{Metric} \to \text{Connection} \to \text{Geodesic} \to \text{Exponential}.$$

Given two points $x, y \in M$ connectable by a geodesic line $\gamma : [0,\ 1] \to M$, the Riemannian distance between such points is defined by

$$d(x, y) \overset{\text{def}}{=} \int_0^1 \mathbb{G}^{\frac{1}{2}}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))dt. \tag{21}$$

A fundamental result of the calculus on manifolds states that the Riemannian gradient of a squared distance function reads:

$$\mathbb{G}^{\sharp}_x(d_x d^2(x, y)) = -2\log_x(y), \tag{22}$$

wherever the logarithm is defined. For some manifolds of interest in applications, endowed with some specific metrics, the explicit expression of the geodesic distance may be unknown in closed form. Likewise, the gradient of the squared distance is known in closed form only if the exponential may be inverted explicitly.

2.2 Second-Order Dynamical Systems on Manifolds

A classical dynamical system on the space $\mathbb{R}^3$ is the Newton's law of motion of a particle of mass m subjected to an external force $f : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{I} \to \mathbb{R}^3$. The trajectory $\gamma : \mathbb{I} \to \mathbb{R}^3$, followed by such a particle, is computed as the solution of the Newton's equation:

$$m\frac{d^2\gamma}{dt^2} = f\left(\gamma, \frac{d\gamma}{dt}, t\right). \tag{23}$$

The term $\ddot{\gamma}(t)$ denotes the instantaneous acceleration of the particle as a function of the time t, the mass term m accounts for the inertia of the particle and the external force f depends on the instantaneous position $\gamma(t)$, on the instantaneous velocity $\dot{\gamma}(t)$ and may depend explicitly on the time t.

A few recent contributions appeared recently in the scientific literature that aim at extending the classical notion of second-order dynamical systems on $\mathbb{R}^n$. In this regard, notable is the contribution [17], that suggests to employ geometric numerical integration techniques to study dynamical systems subjected to both equality and 'hard' inequality constraints. A contribution on the line of extending linear smothering to non-linear (time-independent) stifling on dynamical systems is [18], that introduces the notion of Rayleigh damping.

In the present manuscript, a formulation of dynamical systems on manifold is invoked, which is based on the dynamics of a point-wise particle sliding on a Riemannian manifold M with metric $\mathbb{G}$. The extended stationary-action principle to formulate the dynamics of a particle in a dissipative context is based on the notions of Riemannian kinetic energy function, potential energy function and external driving force.

The kinetic energy function for a point-wise particle associated with the metric $\mathbb{G}$ is denoted by $K : TM \to \mathbb{R}$ and is defined by $K_x(v) \stackrel{\text{def}}{=} \frac{1}{2}\mathbb{G}_x(v, v)$ for $(x, v) \in TM$. On a Riemannian manifold, the metric tensor is positive-definite, hence, on every trajectory $\gamma : \mathbb{I} \to M$, it holds that $K_\gamma(\dot{\gamma}) \geq 0$.

The potential energy function $V : M \to \mathbb{R}$ depends on the coordinate $x \in M$ only. In absence of any external solicitation, the dynamical system generates a trajectory $\gamma : \mathbb{I} \to M$ that follows the landscape of the potential energy function.

In the context of dynamical systems subjected to external driving forces on manifolds, a force at a point $x \in M$ is a cotangent vector, namely $f_x : T_xM \times \mathbb{I} \to T_x^\star M$.

On a Riemannian manifold M whose tangent bundle is endowed with an inner product $\mathbb{G}$, the extended stationary-action principle that governs the evolution of a dynamical system reads:

$$\delta \int_{\mathbb{I}} (K_\gamma(\dot{\gamma}) - V_\gamma) dt + \int_{\mathbb{I}} \mathbb{G}_\gamma(\mathbb{G}_\gamma^\sharp(f_\gamma), \delta\gamma) dt = 0, \tag{24}$$

where the symbol δ denotes variation. The integrand of the leftmost integral represents the classical Lagrangian function of the particle and its integral represents the total action of the particle. The rightmost integral represents the variation of energy due to the external driving force and represents an extension of the classical stationary-action principle[19]. The principle (24) represents a fundamental extension of the Hamiltonian learning/optimization principle introduced in the earlier contributions [20, 21]. On each interior point of the trajectory $t \in \mathbb{I}$, the variation $\delta\gamma \in T_\gamma M$ is arbitrary, while at the boundaries of the trajectory it vanishes to zero[§]. Computing the variation leads to the dynamical system formulation:

$$(\mathbb{G}_\gamma^\flat \circ \overset{\mathbb{G}}{\nabla}_{\dot{\gamma}})\dot{\gamma} = -d_\gamma V_\gamma + f_\gamma. \tag{25}$$

[§]For a reference on the caluslus of variation on smooth manifolds, that justifies the assertion that $\delta\gamma \in T_\gamma M$, readers might consult the paper [22].

By comparing the equation (25) with Newton's law (23), it is readily seen that the term $\overset{\mathbb{G}}{\nabla}_{\dot\gamma}\dot\gamma$ represents the geometric acceleration of the particle sliding on the manifold M, the operator $\mathbb{G}^\flat$ plays the role of inertia tensor and the term $-d_\gamma V_\gamma + f_\gamma$ is the total force that generates the motion of the particle.

The geometric acceleration may be written in terms of the Christoffel form as $\overset{\mathbb{G}}{\nabla}_{\dot\gamma}\dot\gamma = \ddot\gamma + \overset{\mathbb{G}}{\Gamma}_\gamma(\dot\gamma,\dot\gamma)$, therefore, the equation (25) may be rewritten as the system of first-order differential equations:

$$\begin{cases} \dot x = v, \\ \dot v = -\overset{\mathbb{G}}{\Gamma}_x(v,v) - (\mathbb{G}^\sharp_x \circ d_x)V_x + \mathbb{G}^\sharp_x(f_x) \end{cases} \tag{26}$$

in the tangent-bundle variables $(x(t), v(t)) \in TM$.

The total energy (or Hamiltonian function) $H : TM \times \mathbb{I} \to \mathbb{R}$ of the particle sliding on the manifold, and hence of the dynamical system (26), is defined by

$$H_x \overset{\text{def}}{=} K_x + V_x. \tag{27}$$

Calculations show that, over a trajectory $\gamma : \mathbb{I} \to M$ of the system (26), it holds that:

$$\frac{dH_\gamma}{dt} = \mathbb{G}_\gamma(\mathbb{G}^\sharp_\gamma(f_\gamma), \dot\gamma). \tag{28}$$

In the case that the external forcing is absent, the system (26) is conservative as the total energy H_γ keeps constant over time, otherwise, the system is non-conservative and its energy varies over time.

In the present context, the force $f_x = f_x(v,t) \in T^\star_x M$ will essentially represent damping effects and external forcing terms. In particular:

- Friction-type damping: This kind of damping generalizes the Rayleigh damping and is expressed by the forcing term $-\frac{1}{2}\mu\frac{\partial(\mathbb{G}_x(v,v)^\varepsilon)}{\partial v^\sigma}dx^\sigma$, with $\varepsilon \geq 1$ being a damping coefficient and $\mu \geq 0$ being a viscosity coefficient. By computing the derivatives, it is found that the friction-type damping force equals $-\mu\|v\|_x^{2(\varepsilon-1)}\mathbb{G}^\flat_x(v)$.

- Non-linear damping: It generalizes the nonlinear damping term that appears in the Van der Pol system and assumes the expression $-(\mathbb{G}^\flat_x \circ \varphi_x)(v)$, with $\varphi \in \mathfrak{X}(M)$ being a vector field that depends on a parameter, such that, for each $x \in M$, $\varphi_x : T_xM \to T_xM$ is a linear map (endomorphism).

- Sinusoidal driving force: It generalizes the notion of external sinusoidal forcing term in the mono-dimensional dynamical systems. It assumes the expression $A\sin(\Omega t)dx^1$, with $A, \Omega \in \mathbb{R}$ and $t \in \mathbb{I}$.

With the above assumptions, the dynamical system (26) assumes the expression:

$$\begin{cases} \dot x = v, \\ \dot v = -\overset{\mathbb{G}}{\Gamma}_x(v,v) - \mathbb{G}^\sharp_x(d_xV_x) - \mu\|v\|_x^{2(\varepsilon-1)}v \\ \qquad -\varphi_x(v) - A\sin(\Omega t)\mathbb{G}^\sharp_x(dx^1). \end{cases} \tag{29}$$

The rate of change of the total energy due to the above forcing terms reads:

$$\frac{dH_x}{dt} = -\mu\|v\|_x^{2\varepsilon} - \mathbb{G}_x(\varphi_x(v), v) - A\sin(\Omega t)v^1. \tag{30}$$

If $\mu \neq 0$, the first term is purely dissipative and is proportional to the kinetic energy of the system (to K_x^ε, in fact), while the second term on the right-hand side is not necessarily dissipative and may bring energy into the system. If the vector field φ is positive-definite, then the term $-\varphi_x(v)$ in the system (29) is purely dissipative.

In the special case that $M = \mathbb{R}^n$, hence $T_xM \cong \mathbb{R}^n$, and $\mathbb{G}_x(v, w) = v^{\mathrm{T}}w$, it holds that $\overset{\mathbb{G}}{\Gamma}_x = 0$, $\mathbb{G}_x^\sharp(d_xV_x) = (J_V)_x$, $\mathbb{G}_x^\sharp(dx^1) = \mathbf{1}_n$, where symbol $\mathbf{1}_n$ denotes the vector $[1, 0, \cdots, 0]^{\mathrm{T}} \in \mathbb{R}^n$. Hence, the dynamical system (29) becomes:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -(J_V)_x - \mu(v^{\mathrm{T}}v)^{\varepsilon-1}v - \varphi_x(v) - A\sin(\Omega t)\mathbf{1}_n, \end{cases} \tag{31}$$

where $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mu > 0$. The dynamical system (31) accounts for the exemplary systems (2), (4), (5), (6), (7), (11).

2.3 Potential Energy Functions

The present subsection retraces the classical potential energy functions recalled in the Introduction and extends such potentials to a general Riemannian manifold.

A well-documented potential energy function is the one arising in the study of the simple pendulum. In the classical case that $M = \mathbb{R}$, it reads $V_x \propto 1 - \cos x$. Such potential energy function may be extended to a general Riemannian manifold M, endowed with a Riemannian distance function $d(\cdot, \cdot)$, as:

$$V_x^{(\mathrm{pen})} \overset{\mathrm{def}}{=} \kappa(1 - \cos d(x, r)), \tag{32}$$

with $\kappa > 0$ being a constant parameter and $r \in M$ denoting a reference point. The potential $V_x^{(\mathrm{pen})}$ presents one of its minima in $x = r$. Rewrite the pendulum-type potential as $V_x^{(\mathrm{pen})} = \kappa - \kappa\cos[(d^2(x, r))^{\frac{1}{2}}]$. According to the calculation rule (22), its Riemannian gradient reads:

$$\mathbb{G}_x^\sharp(d_xV_x^{(\mathrm{pen})}) = -\kappa\frac{\sin d(x, r)}{d(x, r)}\log_x(r). \tag{33}$$

The potential energy function introduced to state the Van der Pol dynamical system in the case $M = \mathbb{R}$ is a quadratic function, namely $V_x \propto x^2$. Such potential energy function may be extended to a general Riemannian manifold M, endowed with a Riemannian distance function $d(\cdot, \cdot)$, as:

$$V_x^{(\mathrm{pol})} \overset{\mathrm{def}}{=} \frac{1}{2}\kappa d^2(x, r), \tag{34}$$

with $\kappa > 0$ being a constant parameter and $r \in M$ denoting a reference point. The potential $V_x^{(\mathrm{pol})}$ presents its minimum in $x = r$. According to the calculation rule (22), its Riemannian gradient reads:

$$\mathbb{G}_x^\sharp(d_xV_x^{(\mathrm{pol})}) = -\kappa\log_x(r). \tag{35}$$

The contribution [23] studied an extension of the (hard) Duffing oscillator to two dimensions, namely, to the manifold $M = \mathbb{R}^2$. The key point is to extend the potential function $V_x : M \to \mathbb{R}$ as:

$$V_x = \frac{1}{2}\Omega_0^2 \|x\|^2 + \frac{1}{4}\kappa\Omega_0^2 \|x\|^4, \tag{36}$$

where $\|\cdot\|$ denotes the Euclidean norm, and $\Omega_0 > 0$ and $\kappa > 0$ are free parameters. The quantity $\|x\|$ coincides with the Euclidean distance of the state x to 0, namely, it coincides with the Euclidean distance $d^{\mathrm{E}}(x, 0)$. Such observation paves the way to an extension to a general Riemannian manifold M, by replacing the term $d^{\mathrm{E}}(x, 0)$ with the term $d(x, r)$:

$$V_x^{(\mathrm{duf})} \stackrel{\mathrm{def}}{=} \pm\frac{1}{2}d^2(x, r) \pm \frac{1}{4}\kappa d^4(x, r), \tag{37}$$

where again $d(\cdot, \cdot)$ denotes the Riemannian (geodesic) distance on the manifold M, $\kappa > 0$ is a free parameter and $r \in M$ denotes a reference point. The signs $\pm$ were introduced to account for the soft and the double-well Duffing oscillator, together with the hard Duffing oscillator analyzed in [23]. According to the calculation rule (22), its Riemannian gradient, in the case of hard Duffing oscillator, reads:

$$\mathbb{G}_x^\sharp(d_x V_x^{(\mathrm{duf})}) = -[1 + \kappa d^2(x, r)] \log_x(r). \tag{38}$$

Likewise, the Keplerian system mentioned in [23], may be extended to a general Riemannian manifold M with Riemannian distance $d(\cdot, \cdot)$, by defining the Keplerian potential as:

$$V_x^{(\mathrm{kep})} \stackrel{\mathrm{def}}{=} -\frac{\rho}{d(x, r)} + \varepsilon d(x, r), \tag{39}$$

with $\rho, \varepsilon > 0$. Rewriting the Keplerian potential as $-\rho[d^2(x, r)]^{-\frac{1}{2}} + \varepsilon[d^2(x, r)]^{\frac{1}{2}}$ and invoking again the calculation rule (22), its Riemannian gradient was calculated to be

$$\mathbb{G}_x^\sharp(d_x V_x^{(\mathrm{kep})}) = -\left[\frac{\rho}{d^3(x, r)} + \frac{\varepsilon}{d(x, r)}\right] \log_x(r). \tag{40}$$

It is straightforward to envisage that all potential functions known in the literature based on distance functions may be extended to Riemannian manifolds on the basis of the Riemannian distance function. A well known example is the Lennard-Jones potential of computational chemistry or the electric/gravitational potential.

In the case that the manifold M has a special structure, as, for instance, in the case that M is a matrix manifold, customized potential energy functions may be designed, as illustrated in Section 3.

It is interesting to point out that a potential energy term V_x may be also used to couple two dynamical systems. Such observation might constitute the basis for the study of coupled oscillators on manifolds, a topic that will be analyzed in the future.

3 Nonlinear Damped Oscillators on Special Manifolds

The present section illustrates the structure of the general non-linear damped oscillator (26) by recurring to two specific examples. The specific manifolds of interest recalled in the following

subsections, of particular interest in applications, are a Lie-group-type manifold, namely, the manifold of symmetric positive-definite matrices (discussed in Subsection 3.1) and the compact Stiefel manifold (discussed in Subsection 3.2). The following two examples illustrate the main difficulties as well as the positive results concerning the writing of Equation (26) in specific cases.

3.1 Nonlinear Oscillator over the Manifold of Symmetric Positive-Definite Matrices

The manifold of symmetric, positive-definite matrices is defined as the set $\mathrm{S}^+(n) \stackrel{\mathrm{def}}{=} \{x \in \mathbb{R}^{n\times n} | x^{\mathrm{T}} - x = 0,\ x > 0\}$. The tangent bundle exhibits a trivial structure as $T_x\mathrm{S}^+(n) = \mathrm{S}(n) \stackrel{\mathrm{def}}{=} \{v \in \mathbb{R}^{n\times n} | v^{\mathrm{T}} - v = 0\}$ for every $x \in \mathrm{S}^+(n)$. Symmetric positive-definite matrices find a wide range of applications. For instance, symmetric positive-definite matrices are applied in low-rank approximation of correlation matrices[24], in the analysis of deformation[25, 26], in pattern recognition[27], in automatic and intelligent control[28], in the estimation of the power spectrum of random processes[29], in cognitive computation[30] and in computational neurology[31].

The manifold $\mathrm{S}^+(n)$ of symmetric positive-definite matrices, endowed with its canonical metric, has associated the following geometric quantities:

$$\mathbb{G}_x(w, v) = \mathrm{tr}(x^{-1}wx^{-1}v), \tag{41}$$

$$\overset{\mathbb{G}}{\Gamma}_x(v, v) = -vx^{-1}v, \tag{42}$$

$$\exp_x(v) = \sqrt{x}\exp\left(\sqrt{x^{-1}}v\sqrt{x^{-1}}\right)\sqrt{x}, \tag{43}$$

with $x \in \mathrm{S}^+(n)$ and $w, v \in T_x\mathrm{S}^+(n)$. The symbol $\sqrt{\cdot}$ denotes the symmetric matrix square root, while the matrix-to-matrix operator $\exp(\cdot)$ denotes matrix exponential. The expression of the exponential map (43) may be easily inverted and it holds that:

$$\log_x(y) = \sqrt{x}\log\left(\sqrt{x^{-1}}y\sqrt{x^{-1}}\right)\sqrt{x}, \tag{44}$$

where the matrix-to-matrix operator $\log(\cdot)$ denotes (principal) matrix logarithm.

If the potential energy function V_x is written in terms of geodesic distance, its Riemannian gradient may be written by invoking the property (22). If the potential energy function is written as an explicit function of the matrix-variable x, then the following expression of its Riemannian gradient, in terms of its Jacobian matrix $(J_V)_x$ of partial derivatives with respect to the coordinates in x, is available:

$$\mathbb{G}^\sharp_x(d_xV_x) = \frac{1}{2}x\left((J_V)_x + (J_V)_x^{\mathrm{T}}\right)x. \tag{45}$$

By gathering the expressions of the Christoffel operator and of the Riemannian gradient of the potential energy function, the following dynamical system equations are obtained:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = vx^{-1}v - \mathbb{G}^\sharp_x(d_xV_x) - \mu\|v\|_x^{2(\varepsilon-1)}v - \varphi_x(v) \\ \qquad -A\sin(\Omega t)x\left(\mathbf{1}^{\mathrm{T}}_{n\times n} + \mathbf{1}_{n\times n}\right)x. \end{cases} \tag{46}$$

In the present case, $\varphi_x : \mathrm{S}(n) \to \mathrm{S}(n)$. The flow corresponding to the second differential equation of the system (46) takes place on a trivial tangent bundle.

3.2 Nonlinear Oscillator over the Compact Stiefel Manifold

The compact Stiefel manifold is defined as $\mathrm{St}(n,p) \stackrel{\mathrm{def}}{=} \{x \in \mathbb{R}^{n\times p} | x^{\mathrm{T}}x = e_p\}$, where $p \leq n$. The tangent spaces exhibit the structure $T_x\mathrm{St}(n,p) \stackrel{\mathrm{def}}{=} \{v \in \mathbb{R}^{n\times p} | x^{\mathrm{T}}v + v^{\mathrm{T}}x = 0\}$. Exemplary applications where the compact Stiefel manifold plays a prominent role are blind source separation upon signal pre-whitening and independent component analysis[32–34], non-negative matrix factorization[35], direction of arrival estimation[36], best basis search/selection[37, 38], electronic structures computation within local density approximation, e.g., for understanding the thermodynamics of bulk materials, the structure and dynamics of surfaces, and the nature of point-defects in crystals[39] and factor analysis in psychometrics[40].

When the Stiefel manifold is endowed with its canonical metric, the following geometric characterization holds:

$$\mathbb{G}_x(w,v) = \mathrm{tr}\left(w^{\mathrm{T}}v\right) - \frac{1}{2}\mathrm{tr}\left(w^{\mathrm{T}}xx^{\mathrm{T}}v\right), \tag{47}$$

$$\overset{\mathbb{G}}{\Gamma}_x(v,v) = -vv^{\mathrm{T}}x - xv^{\mathrm{T}}(e_n - xx^{\mathrm{T}})v, \tag{48}$$

$$\exp_x(v) = [x\ q]\exp\left(\begin{bmatrix} x^{\mathrm{T}}v & -r^{\mathrm{T}} \\ r & 0_p \end{bmatrix}\right)\begin{bmatrix} e_p \\ 0_p \end{bmatrix}, \tag{49}$$

$$\mathbb{G}^{\sharp}_x(d_xV_x) = (J_V)_x - x(J_V)_x^{\mathrm{T}}x, \tag{50}$$

where q and r denote the factors of the compact QR decomposition of the matrix $(e_n - xx^{\mathrm{T}})v$ and 0_p denotes a zero $p \times p$ matrix. The closed-form expression of the logarithmic function related with the exponential function (49) is unknown[41] (except for the cases $p = 1$ and $p = n$). Consequently, the expression (50) of the Riemannian gradient of a potential energy function in terms of its Jacobian is offered. It proves useful in those applications where the potential energy function V_x is written explicitly in terms of the matrix-variable x as, for example, in the case of principal/independent component analysis. In the principal/minor component analysis case, for instance, the potential energy function reads $V_x = \pm\frac{1}{2}\mathrm{tr}(x^{\mathrm{T}}\Sigma x)$, with $\Sigma \in \mathrm{S}^+(n)$, and its Jacobian matrix reads $(J_V)_x = \pm\Sigma x$.

By the expressions of the Riemannian gradient of the potential energy function and of the Christoffel form, it is obtained the dynamical system:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -vv^{\mathrm{T}}x - xv^{\mathrm{T}}(e_n - xx^{\mathrm{T}})v \\ \qquad -((J_V)_x - x(J_V)_x^{\mathrm{T}}x) - \mu\|v\|_x^{2(\varepsilon-1)}v \\ \qquad -\varphi_x(v) - A\sin(\Omega t)(\mathbf{1}_{n\times p} - x\mathbf{1}_{n\times p}^{\mathrm{T}}x), \end{cases} \tag{51}$$

where it must hold that $\varphi_x : T_x\mathrm{St}(n,p) \to T_x\mathrm{St}(n,p)$.

4 Conclusion

The present manuscript illustrates the state of the art and the current knowledge about the formulation of a nonlinear damped oscillator theory by stationary-action principle. In particular, the present contribution recalled the notation and some notion from differential geometry that are instrumental in the development of a second-order dynamical-system theory on differentiable (and, specifically, Riemannian) manifolds and sets out the fundamental principles for defining a non-linear, actively/passively damped oscillator theory on curved manifolds.

The motivation and fundamental aim of the present contribution was to open new perspectives in the theory of nonlinear damped oscillators and to promote research efforts in this field.

A fundamental aspect of the present research concerns the implementation on a computer platform of the discussed non-linear damped oscillator. The numerical integration of the dynamical system equations (26) is nontrivial and needs invoking specific numeric techniques pertaining to the field of geometric numeric integration. Such fundamental aspect needs specific knowledge, that ought to be recalled from the scientific literature, and is rich in results, hence it is too hefty to become merely a part of the present paper. A contribution that specifically illustrates the numerical implementation aspects of the proposed non-linear damped oscillator theory is the logical continuation of the present research.

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