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RESEARCH ARTICLE

Simone FIORI

Averaging over the Lie Group of Optical Systems Transference Matrices

Abstract Motivated by recent studies in ophthalmology about the computation of the average character of a set of eyes, the present manuscript discusses an iterative approach to tackle the problem of computing the average matrix out of a set of optical-system-transference matrices. The manuscript recalls how the optical features of a human eye may be described in terms of a linear optical model. The paper also shows that the linear optical models form a Lie group under appropriate algebraic and geometric settings. An averaging algorithm that copes with Lie groups is recalled from the scientific literature and described. The present paper also recalls a previous non-iterative averaging rule by Harris [8]. Numerical results about the averaging of optical transference matrices suggest that Harris' rule is nearly optimal, as Lie-group averaging iteration can improve Harris' result only slightly.

Keywords Average eye, Lie-group theory, real symplectic group, optical transference operators.

Simone FIORI ( )

Dipartimento di Ingegneria Biomedica, Elettronica e Telecomunicazioni (DiBET), Facoltà di Ingegneria, Università Politecnica delle Marche, Via Brecce Bianche, Ancona I-60131, Italy URL: web.dibet.univpm.it/fiori – E-mail: s.fiori@univpm.it

1 Introduction

In the study of optical systems in ophthalmology, it is assumed that the first-order features of an optical system are described by a linear operator $\tau \in \mathbb{R}^{5\times 5}$ termed 'optical transference matrix' [7, 8]. In the most general case, a transference matrix has the form:

$$\tau \stackrel{\text{def}}{=} \begin{bmatrix} s & \delta \\ 0_{4,1}^T & 1 \end{bmatrix}, \tag{1}$$

where $\delta \in \mathbb{R}^4$, symbol $0_{4,1} \in \mathbb{R}^4$ denotes an all-zero vector, the submatrix s is real symplectic, namely, it belongs to the set $\mathbb{S}(4,\mathbb{R})$ (see Section 3.1) and superscript ^{T} denotes matrix transpose.

Transference matrices provide a way of tracing a ray through a compound optical system [7]. Descriptions based on transference matrices are quite general, as they allow to describe coaxial spherical systems as well as astigmatic surfaces, non-coaxial systems and systems containing prisms and decentered lenses. A compound optical system may be described, through a law of composition, via the transference matrices of its subsystems. A typical example is the case of an eye, that may be described in terms of six optical subsystems, namely, the cornea, the anterior chamber, the anterior surface of the lens, the interior of the lens, the posterior surface of the lens and the gap between the lens and the retina. Such a first-order characterization of an eye provides a description of how an eye modifies a ray of light traversing it, namely, how an eye changes the state of the ray at incidence into the state of the ray at retina.

A question that arose in ophthalmology is how to define and calculate a meaningful average optical system whose optical character is an average of the optical character of the systems in a set. Although the problem is formulated for a general linear optical system, its main motivation is the computation of average models of human eyes [8]. The problem of the computational modeling of the human eye has been widely studied in different scientific and technological fields. One of the main reasons of such increasing interest is the possibility of reproducing the optical properties of the eye by means of computational simulations. Eye models make it possible to develop efficient devices to treat and to correct problems affecting the vision [2, 11]. For instance, laser ablation of the front surface of the cornea, a surgical technique used to correct the refractive errors of the eye, modifies the optical features of the first subsystem in the eye's compound optical model. The whole model hence predicts the global effects of ablation surgery on a patient's vision ability. Averaging provides a statistical analysis of the effects of ablation surgery and, in turn, provides a guidance about the average depth of ablation. As another example, optical model averaging might be applicable as a part of the modern techniques of design of soft contact lenses. In particular, averaging affords contact-lens design that can take into account the features of similar patient's eyes, adapting the surface of contact lenses to match the average optical features of a category of patients.

Over a set of N optical systems, whose optical transference matrices are denoted by τ_n, $n = 1, \ldots, N$, the average character of the systems may be defined as an average of the transference matrices τ_n that preserves the transference-matrix structure (1) (see [8] and references therein). It is clear that such a solution may not be the arithmetic mean:

$$\frac{1}{N}(\tau_1 + \tau_2 + \cdots + \tau_N),$$

because, in general, summation does not preserve symplecticity. Likewise, the geometric mean:

$$(\tau_1 \cdot \tau_2 \cdots \tau_N)^{\frac{1}{N}},$$

is not acceptable because it actually depends on the order of computation of matrix products.

In the paper [8], Harris proposed a possible scheme to compute the average of a set of optical system transference matrices, termed 'exponential-mean-log'. The present manuscript tackles the problem of estimating an empirical ensemble average of a set of optical transference matrices within the more general framework of Lie-group elements averaging. The paper describes the Lie-group structure of the space of optical transference operators and recalls the general averaging scheme for Lie groups recently proposed in [3]. The paper then applies the general scheme to the case of transference Lie group.

2 Optical system modeling by transference matrices

Eyes are organs that detect light and send electrical impulses along the optic nerve to the visual area as well as other areas of the brain. An eyeball, as shown in the Figure 1, may be modeled as an optical system formed by the following components:

1. *Cornea*, which is the transparent front part of the eye that covers the iris, pupil, and anterior chamber. Together with the lens, the cornea refracts light, accounting for approximately two-thirds of the eye's total optical power;

2. *Anterior chamber*, filled-up of a fluid termed *aqueous humour*;

3. *Anterior surface of the lens*;

4. *Interior of lens*, which is an optical element that transmits and refracts light, converging or diverging the incident ray of light;

5. *Posterior surface of the lens*;

6. *Gap between the posterior surface of the lens and the retina*, filled-up of a fluid termed *vitreous humour*;

7. *Retina*, which is a light-sensitive tissue lining the inner surface of the eye. The optics of the eye create an image of the visual world on the retina. Light striking the retina initiates a cascade of chemical and electrical events that ultimately trigger nerve impulses.

The above optical elements may be coaxial as well as non-coaxial. As an example of non-coaxial case, the lens may move off-axis as a result of a mechanical trauma. (In this event, the ray of light traversing the eye impacts the retina in a wrong place and it is necessary to rotate the eyeball to compensate for the resulting image-distortion effect.)

From a linear optics perspective, a ray of light may be represented through its augmented ray state, described by vector:

$$\rho = \begin{bmatrix} \alpha \\ y \\ 1 \end{bmatrix}, \tag{2}$$

where vector $\alpha \in \mathbb{R}^2$ is the reduced slope vector of the ray and vector $y \in \mathbb{R}^2$ denotes the position at which a ray enters or leaves an optical system. If ρ_{in} denotes the state of a ray

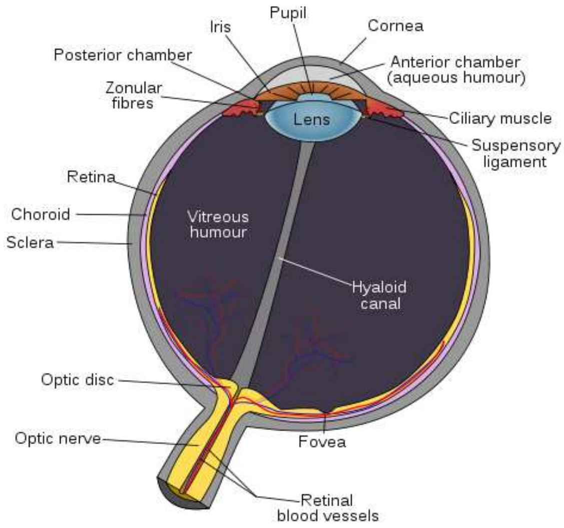

Figure 1: Schematic of a human eyeball (Source: Wikipedia [12]).

entering an optical system and ρ_{out} denotes the state of a ray leaving the same optical system, their relationship expresses as:

$$\rho_{\text{out}} = \tau\rho_{\text{in}}, \tag{3}$$

where τ denotes the transference matrix of the optical system. A ray of light that enters an optical system with state ρ_{in} and leaves the optical system with state ρ_{out} is shown in the Figure 2. Any transference matrix has the form (1), namely:

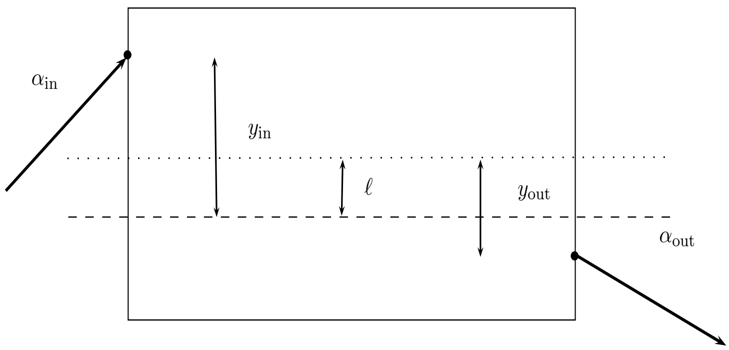

Figure 2: Augmented state-vectors of rays of light entering/leaving an optical system. Some optical parts may be decentered.

$$\tau \stackrel{\text{def}}{=} \begin{bmatrix} s & \delta \\ 0_{4,1}^T & 1 \end{bmatrix}, \tag{4}$$

where the astigmatic system submatrix s describes the diffractive power of the optical system and vector δ describes the displacement of the emerging ray with respect to the incident ray of light.

The eyeball is formed of elements that can either be modeled as thin lens, refracting interfaces or thin homogeneous media, as shown in the Figure 3.

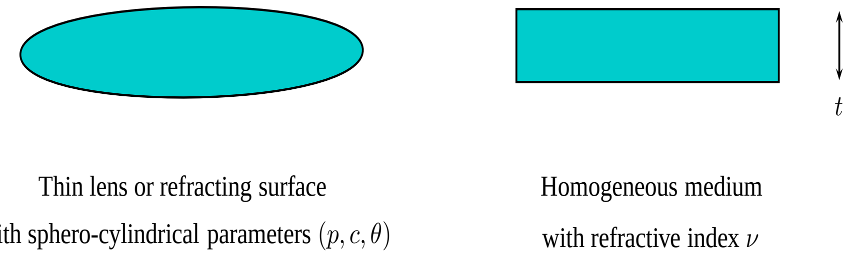

Figure 3: Types of thin optical elements.

For a thin lens or a single refracting interface, the system matrix takes on the form [6, 10]:

$$s \stackrel{\text{def}}{=} \begin{bmatrix} e_2 & -f \\ 0_2 & e_2 \end{bmatrix}, \; f \stackrel{\text{def}}{=} \begin{bmatrix} p + c\sin^2(\theta) & -\frac{1}{2}c\sin(2\theta) \\ -\frac{1}{2}c\sin(2\theta) & p + c\cos^2(\theta) \end{bmatrix}, \tag{5}$$

where symbol e_2 denotes a 2×2 identity matrix, symbol 0_2 denotes a 2×2 all-zero matrix, $f \in \mathbb{R}^{2\times 2}$ is termed *dioptric power matrix* of the optical system and (p, c, θ) are termed *sphero-cylindrical parameters* of the thin optical system. The dioptric power matrix measures in dioptres.

For an elementary optical system in which a ray traverses a thin homogeneous medium, the system matrix takes on the form [6]:

$$s \stackrel{\text{def}}{=} \begin{bmatrix} e_2 & 0_2 \\ \frac{t}{\nu}e_2 & e_2 \end{bmatrix}, \tag{6}$$

where $t \in \mathbb{R}^+$ denotes the thickness of the homogeneous medium and $\nu \in \mathbb{R}^+$ denotes its refractive index. The reduced thickness $\frac{t}{\nu}$ (also termed *optical thickness* [1]) measures in meters.

For all thin systems, the system vector δ has the form [6]:

$$\delta = \begin{bmatrix} \pi \\ 0_{2,1} \end{bmatrix}, \tag{7}$$

where $\pi \in \mathbb{R}^2$ is termed *prismatic power* and $0_{2,1} \in \mathbb{R}^2$ denotes an all-zero vector. There exist several cases of non-coaxial systems, such as decentered lenses, tilted plane interfaces and prisms. For a decentered lens' refracting surface of optical power f, whose off-axis position in the horizontal and in the vertical directions is described by the displacement vector $\ell \in \mathbb{R}^2$ (see again Figure 2), the prismatic power may be expressed as [6]:

$$\pi = f\ell. \tag{8}$$

For a centered system, it holds $\pi = 0_{2,1}$.

As a ray of light incident on the cornea traverses the eyeball making its way to the retina through six optical elements (cornea, anterior chamber, anterior surface of the lens, interior of the lens, posterior surface of the lens, gap between the lens and the retina), the optical transference matrix of a model eye may be described as:

$$\tau = \tau_6\tau_5\tau_4\tau_3\tau_2\tau_1, \tag{9}$$

where:

- the transference matrix τ_1 describes the optical behavior of the *cornea*. Its system matrix s_1 may be calculated using equation (5), while its system vector δ_1 equals zero;

- the transference matrix τ_2 describes the optical behavior of the *anterior chamber*. Its system matrix s_2 may be calculated using equation (6), while its system vector δ_2 equals zero. The refractive index, in this case, describes the refraction properties of the *acqueous humour*;

- the transference matrix τ_3 describes the optical behavior of the *anterior surface of the lens*. Its system matrix s_3 may be calculated using equation (5). Its system vector δ_3 equals zero if the lens is centered while it may be calculated by using equations (7)+(8) if the lens is decentered;

- the transference matrix τ_4 describes the optical behavior of the *lens*. Its system matrix s_4 may be calculated using equation (6), while its system vector δ_4 equals zero. The refractive index, in this case, describes the refraction properties of the lens;

- the transference matrix τ_5 describes the optical behavior of the *posterior surface of the lens*. Its system matrix s_5 may be calculated using equation (5). Its system vector δ_5 equals zero if the lens is centered while it may be calculated by using equations (7) and (8) if the lens is decentered;

- the transference matrix τ_6 describes the optical behavior of the *gap between the posterior surface of the lens and the retina*. Its system matrix s_6 may be calculated using equation (6), while its system vector δ_6 equals zero. The refractive index, in this case, describes the refraction properties of the *vitreous humour*.

Numerical examples of calculation may be find, e.g., in [6]. Typical values of the sphero-cylindrical parameters as well as of the thickness and refractive index of eyeball model's elements may be found, e.g., in [5, 10]. Values of distances and refractive indices pertaining to the standard Gullstrand's model are shown in the Figure 4. Mathematical eye models using

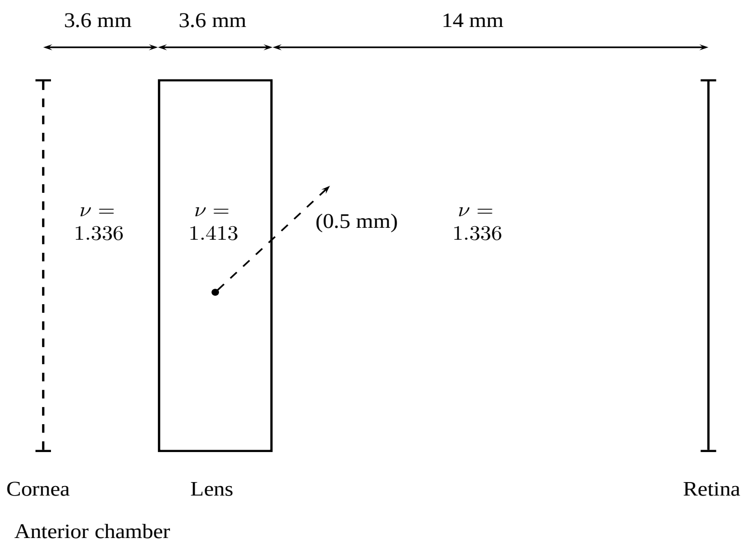

Figure 4: Gullstrand eyeball model: Typical values of the distances among elementary optical subsystems in a human eyeball, of the refractive index and of the absolute displacement of the lens.

Gullstrand's eye model modified by clinically-measured data are also available in the specific scientific literature. For instance, a Gullstrand's eye model modified by clinically-measured data to study human eye aberrations and their compensation for high-resolution retinal imaging is available in [13].

3 Averaging over the transference Lie group

The present section describes the space of optical transference matrices and shows that it has the structure of a Lie group under appropriate group operations. Moreover, the present section summarizes an algorithm for mean-matrix computation on Lie groups and explains how such algorithm can be applied to the case of averaging over the space of optical transference matrices.

3.1 Symplectic group and optical transference group

A space of interest in the theory of optical transference matrices is the real symplectic group:

$$\mathbb{S}(2p,\mathbb{R})\stackrel{\text{def}}{=}\{s\in\mathbb{R}^{2p\times 2p}|s^Tqs=q\},\ q\stackrel{\text{def}}{=}\begin{bmatrix}0_p & e_p\\ -e_p & 0_p\end{bmatrix},\tag{10}$$

where symbol e_p denotes the $p\times p$ identity matrix, symbol 0_p denotes a whole-zero $p\times p$ matrix and the matrix q is termed *fundamental skew-symmetric matrix*. The fundamental skew-symmetric matrix enjoys the following properties: $q^2=-e_{2p}$, $q^{-1}=q^T=-q$.

The space $\mathbb{S}(2p,\mathbb{R})$ is a curved smooth manifold that may also be endowed with algebraic-group operations (namely, group multiplication and group inverse) in a manner that is compatible with the manifold structure and that possesses an identity element. Therefore, the space $\mathbb{S}(2p,\mathbb{R})$ has the structure of a Lie group.

In particular, standard matrix multiplication and inverse work as algebraic group operations. For the matrix multiplication, the property may be proven by noting that, for every $s_1,s_2\in\mathbb{S}(2p,\mathbb{R})$, it holds $s_1s_2\in\mathbb{S}(2p,\mathbb{R})$, in fact:

$$(s_1s_2)^Tq(s_1s_2)=s_2^T(s_1^Tqs_1)s_2=s_2^Tqs_2=q.$$

Any symplectic matrix $s\in\mathbb{S}(2p,\mathbb{R})$ is such that $\det^2(s)=1$, where operator $\det(\cdot)$ denotes determinant. In fact, from the property $s^Tqs=q$ it follows that $\det(s)\det(q)\det(s)=\det(q)$.

An implication of the above property is that any symplectic matrix is invertible. If $s \in \mathbb{S}(2p, \mathbb{R})$ then also $s^{-1} \in \mathbb{S}(2p, \mathbb{R})$. Such property may be proven by negation, namely, by trying to prove that $(s^{-1})^T q s^{-1} \neq q$. If it would be the case, it would hold:

$$(s^{-1})^T q s^{-1} \neq q \;\Rightarrow\; (s^T)^{-1} q s^{-1} \neq q \;\Rightarrow\; q \neq s^T q s,$$

that would contradict the hypothesis that $s \in \mathbb{S}(2p, \mathbb{R})$. The identity element of the group $\mathbb{S}(2p, \mathbb{R})$ is clearly the matrix e_{2p}.

The structure of the tangent space to the manifold $\mathbb{S}(2p, \mathbb{R})$ at every point $s \in \mathbb{S}(2p, \mathbb{R})$, denoted as $T_s\mathbb{S}(2p, \mathbb{R})$, may be uncovered by the help of smooth curves $c : [-a\ a] \to \mathbb{S}(2p, \mathbb{R})$, $a > 0$, such that $c(0) = s$. The vector $v \stackrel{\text{def}}{=} \dot{c}(0)$ belongs to $T_s\mathbb{S}(2p, \mathbb{R})$. As the curve $c(t)$ must satisfy:

$$c^T(t) q c(t) = q,\ \forall t \in [-a\ a], \tag{11}$$

then it should hold:

$$\dot{c}^T(t) q c(t) + c^T(t) q \dot{c}(t) = 0_{2p},\ \forall t \in [-a\ a] \tag{12}$$

and, in particular, for $t = 0$. The tangent space $T_s\mathbb{S}(2p, \mathbb{R})$ has thus the structure:

$$T_s\mathbb{S}(2p, \mathbb{R}) = \{v \in \mathbb{R}^{2p\times 2p} | v^T q s + s^T q v = 0_{2p}\}. \tag{13}$$

In particular, the tangent space at the identity of the Lie group, namely the Lie algebra $\mathbb{H}(2p, \mathbb{R}) \stackrel{\text{def}}{=} T_{e_{2p}}\mathbb{S}(2p, \mathbb{R})$, has the structure:

$$\mathbb{H}(2p, \mathbb{R}) = \{h \in \mathbb{R}^{2p\times 2p} | h^T q + q h = 0_{2p}\}, \tag{14}$$

namely, it coincides with the space of $2p \times 2p$ Hamiltonian matrices. It is worth noting that if $h \in \mathbb{H}(2p, \mathbb{R})$, then $qh - (qh)^T = 0_{2p}$, namely a matrix h is Hamiltonian if and only if the product qh is symmetric, therefore, any Hamiltonian matrix may be parameterized by a symmetric matrix via multiplication by the fundamental skew-symmetric matrix q.

The space of matrices of the kind (1), namely the space of optical transference matrices, is defined as:

$$\mathbb{E} \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} s & \delta \\ 0_{4,1}^T & 1 \end{bmatrix} \middle| s \in \mathbb{S}(4, \mathbb{R}),\ \delta \in \mathbb{R}^4 \right\}. \tag{15}$$

Apparently, the space $\mathbb{E}$ may be endowed with the structure of an algebraic group by standard matrix multiplication and inverse. About multiplication, if $\tau_1, \tau_2 \in \mathbb{E}$, it holds:

$$\tau_1\tau_2 = \begin{bmatrix} s_1 & \delta_1 \\ 0_{4,1}^T & 1 \end{bmatrix} \begin{bmatrix} s_2 & \delta_2 \\ 0_{4,1}^T & 1 \end{bmatrix} = \begin{bmatrix} s_1 s_2 & \delta_1 + s_1\delta_2 \\ 0_{4,1}^T & 1 \end{bmatrix}, \tag{16}$$

thus $\tau_1\tau_2 \in \mathbb{E}$. Given the general form of a transference matrix $\tau \in \mathbb{E}$, it is found that $\det^2(\tau) = \det^2(s)$, therefore, any transference matrix is invertible. Moreover, about the inverse operator, if $\tau \in \mathbb{E}$, it holds:

$$\tau^{-1} = \begin{bmatrix} s^{-1} & -s^{-1}\delta \\ 0_{4,1}^T & 1 \end{bmatrix}, \tag{17}$$

therefore $\tau^{-1} \in \mathbb{E}$. The identity element in $\mathbb{E}$ with respect to matrix multiplication and inverse is apparently the identity matrix e_5.

The optical transference group $\mathbb{E}$ is isomorphic to $\mathbb{S}(4, \mathbb{R}) \times \mathbb{R}^4$. As a product of two differential manifolds, it is a differential manifold as well. As a group manifold, the optical transference group $\mathbb{E}$ is a Lie group.

The tangent space to the set of optical transference $\mathbb{E}$ in a point $\tau = \left[\begin{smallmatrix} s & \delta \\ 0_{4,1}^T & 1 \end{smallmatrix}\right] \in \mathbb{E}$ may be described as follows:

$$T_\tau\mathbb{E} = \left\{ \begin{bmatrix} v & r \\ 0_{4,1}^T & 0 \end{bmatrix} \middle| \, v \in T_s\mathbb{S}(4, \mathbb{R}),\ r \in \mathbb{R}^4 \right\}. \tag{18}$$

On the basis of the above settings, it is possible to develop an averaging algorithm that returns the mean-optical-transference matrix from a set of optical transference matrices.

3.2 Averaging over Lie groups

A notion of differential geometry that is instrumental in the present paper is the one of *retraction map*. Given a differential manifold $\mathbb{G}$ with tangent bundle $T\mathbb{G}$, a retraction $R : T\mathbb{G} \to \mathbb{G}$ maps a tangent vector to the manifold itself. Namely, given a point $\tau \in \mathbb{G}$ and a vector $w \in T_\tau\mathbb{G}$, the quantity $R_\tau(w) \in \mathbb{G}$ (in particular, the element $R_\tau(w)$ belongs to a neighborhood of the element $\tau \in \mathbb{G}$). For the technical details on retractions, please see, e.g., reference [3]. The inverse map of a retraction is termed *lifting map* and is denoted as R_τ^{-1}. A lifting map

R_τ^{-1} is defined only locally on a neighborhood of the point $\tau \in \mathbb{G}$. The notion of retraction map is rendered by the Figure 5.

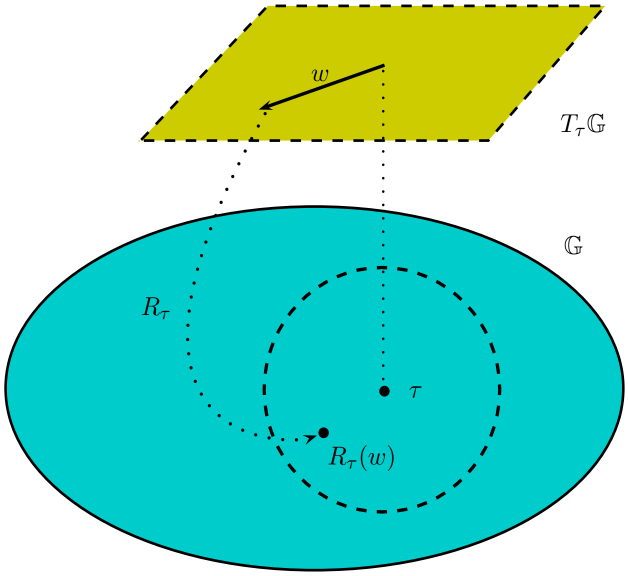

Figure 5: Rendering of the notion of manifold retraction. The point $R_\tau(w) \in \mathbb{G}$ belongs to a neighborhood of the point $\tau \in \mathbb{G}$ denoted by the dashed oval curve.

The paper [3] proposed an iterative algorithm to compute the empirical mean $\mu \in \mathbb{G}$ over a Lie group $\mathbb{G}$. For a matrix Lie group $\mathbb{G}$ whose group structure involves standard matrix multiplication and inverse, the iterative averaging algorithm for a set of N matrices $\tau_n \in \mathbb{G}$ may be expressed as:

$$\mu_{k+1} = R_{\mu_k}\left(\alpha \sum_{n=1}^{N} R_{\mu_k}^{-1}(\tau_n)\right),\ k = 0,\ 1,\ 2,\ \ldots, \tag{19}$$

where constant $\alpha > 0$ plays the role of a relaxation constant that controls the speed and the accuracy of learning. The algorithm (19) generates a sequence of matrices $\mu_k \in \mathbb{G}$ that asymptotically converges to an empirical ensemble average of the matrices τ_n. Paper [3] showed that the value $\alpha = \frac{1}{N}$ is optimal in an algebraic sense, however, smaller values may be used to improve accuracy and convergence. The notion of averaging over a Lie group is rendered by the Figure 6.

For a given Lie group of interest, a retraction/lifting map pair is necessary to set up the averaging algorithm (19). In addition, to start iteration in (19), it is necessary to fix a suitable

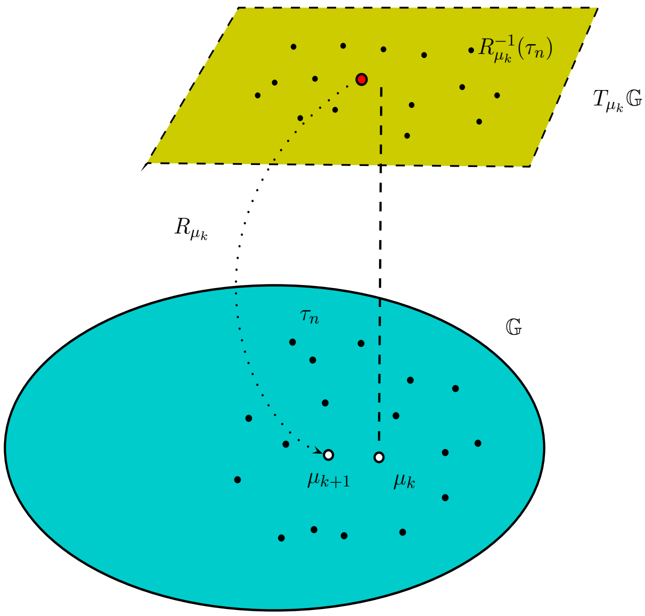

Figure 6: Rendering of the notion of averaging over a Lie group. The red circle on the tangent space $T_{\mu_k}\mathbb{G}$ denotes the vector $\alpha \sum_{n=1}^{N} R^{-1}_{\mu_k}(\tau_n)$.

initial guess $\mu_0 \in \mathbb{E}$.

3.3 A retraction/lifting map pair for the transference Lie group

Let a transference $\tau \in \mathbb{E}$ and a tangent vector $w \in T_\tau\mathbb{E}$ be assigned. The following map is a retraction for the manifold $\mathbb{E}$:

$$R_\tau(w) \stackrel{\text{def}}{=} \tau \exp(\tau^{-1} w), \tag{20}$$

where operator 'exp' denotes the matrix exponential defined by the convergent series:

$$\exp(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{x^k}{k!},\ x \in \mathbb{R}^{5\times 5}. \tag{21}$$

The associated lifting map is defined as:

$$R^{-1}_\tau(\theta) = \tau \log(\tau^{-1}\theta),\ \theta \in \mathbb{E}, \tag{22}$$

where operator 'log' denotes the matrix logarithm defined by the series:

$$\log(x) \stackrel{\text{def}}{=} -\sum_{k=1}^{\infty} \frac{(e_5 - x)^k}{k},\ x \in \mathbb{R}^{5\times 5}. \tag{23}$$

The above series converges on the ball $\{x \in \mathbb{R}^{5\times 5} | \|x - e_5\| < 1\}$, therefore, the lifting map (22) is defined only locally, for θ sufficiently close to τ.

In order to prove that the map (20) is actually a retraction map over the manifold $\mathbb{E}$, take $\tau \in \mathbb{E}$ and $w \in T_\tau\mathbb{E}$ and their block-decompositions:

$$\tau = \begin{bmatrix} s & \delta \\ 0_{4,1}^T & 1 \end{bmatrix}, \; w = \begin{bmatrix} v & r \\ 0_{4,1}^T & 0 \end{bmatrix}. \tag{24}$$

In addition, define functions:

$$t \mapsto R_\tau(tw) = \tau \exp(t\tau^{-1}w), \; t \mapsto F_h(t) \stackrel{\text{def}}{=} \int \exp(th)dt, \tag{25}$$

for $h \in \mathbb{H}(4, \mathbb{R})$. The function $R_\tau(tw)$ takes on values:

$$R_\tau(tw) =$$
$$\begin{bmatrix} s & \delta \\ 0_{4,1}^T & 1 \end{bmatrix} \exp\left(t \begin{bmatrix} s & \delta \\ 0_{4,1}^T & 1 \end{bmatrix}^{-1} \begin{bmatrix} v & r \\ 0_{4,1}^T & 0 \end{bmatrix}\right) =$$
$$\begin{bmatrix} s & \delta \\ 0_{4,1}^T & 1 \end{bmatrix} \sum_{k=0}^\infty \frac{t^k}{k!} \begin{bmatrix} s^{-1}v & s^{-1}r \\ 0_{4,1}^T & 0 \end{bmatrix}^k.$$

Straightforward calculations show that:

$$\begin{bmatrix} s^{-1}v & s^{-1}r \\ 0_{4,1}^T & 0 \end{bmatrix}^0 = e_5,$$
$$\begin{bmatrix} s^{-1}v & s^{-1}r \\ 0_{4,1}^T & 0 \end{bmatrix}^k = \begin{bmatrix} (s^{-1}v)^k & (s^{-1}v)^{k-1}s^{-1}r \\ 0_{4,1}^T & 0 \end{bmatrix},$$

for $k \geq 1$. Hence, it holds:

$$R_\tau(tw) = \begin{bmatrix} s\exp(ts^{-1}v) & \delta + sF_{s^{-1}v}(t)s^{-1}r \\ 0_{4,1}^T & 1 \end{bmatrix}. \tag{26}$$

Now, the expression $s\exp(ts^{-1}v)$ denotes a geodesic arc over the manifold $\mathbb{S}(4, \mathbb{R})$ under the Khevdelidze-Mladenov metrics [4], therefore the map $(s, v) \to s\exp(s^{-1}v)$ is a retraction. Moreover, $\delta + sF_{s^{-1}v}(t)s^{-1}r \in \mathbb{R}^4$. This proves that the map $(s, v) \to R_\tau(w)$ is a retraction.

To prove that the map R_τ^{-1} is a lifting map about $\tau \in \mathbb{E}$, it suffices to note that $R_\tau^{-1}(R_\tau(w)) = w$ whenever $R_\tau(w) \in \mathbb{D}$.

The iterative averaging algorithm (19) implemented with the retraction map (20), takes on the form:

$$\begin{aligned} \mu_{k+1} &= \mu_k \exp\left(\alpha \mu_k^{-1} \sum_{n=1}^{N} \mu_k \log(\mu_k^{-1}\tau_n)\right), \\ &= \mu_k \exp\left(\alpha \sum_{n=1}^{N} \log(\mu_k^{-1}\tau_n)\right), \end{aligned}$$

for $k = 0,\ 1,\ 2,\ \ldots$. The initial guess $\mu_0 \in \mathbb{E}$ should be properly set in order to guarantee fast convergence of the algorithm.

3.4 Harris 'exponential-mean-log' transference

In the paper [8], Harris proposed an 'exponential-mean-log' scheme to compute the average $\mu_{\mathrm{H}} \in \mathbb{E}$ of a set of optical system transference matrices $\tau_n \in \mathbb{E}$ that may be expressed as:

$$\mu_{\mathrm{H}} \stackrel{\mathrm{def}}{=} \exp\left(\frac{1}{N}\sum_{n=1}^{N} \log \tau_n\right). \tag{27}$$

Harris conjectured that the above averaging scheme is consistent, namely, it returns a transference matrix as result. In the subsequent paper [9], Harris and Cardoso published a detailed investigation of the result of the application of the logarithm and exponential operators in the expression (27) and concluded that such averaging expression is indeed consistent.

Numerical experiments show that Harris' exponential-mean-log empirical ensemble average estimate is rather accurate and the iterative algorithm (27) may be used to refine the empirical ensemble average estimate by setting $\mu_0 = \mu_{\mathrm{H}}$.

4 Numerical experiments

In the present section, the behavior of the proposed averaging algorithm over the Lie group of optical transference matrices is illustrated via examples and numerical experiments. In all the following experiments, the relaxation constant $\alpha = \frac{1}{8N}$ was used, as it was found experimentally to guarantee steady and sufficiently accurate convergence.

In order to monitor the convergence rate of the algorithm (27) during iteration, the following average Frobenius scattering index $\varphi : \mathbb{E} \to \mathbb{R}_0^+$ is used:

$$\varphi(\tau) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n=1}^{N} \|\tau - \tau_n\|^2, \tag{28}$$

where symbol $\|\cdot\|$ denotes Frobenius norm. The quantity $\varphi(\mu_k)$ should be decreasing over iterations and it may be compared with the (initial) value $\varphi(\mu_0)$.

4.1 Numerical experiments on randomly generated optical system transference matrices

In order to test numerically the averaging algorithm (27), N random matrices $\tau_n \in \mathbb{E}$ may be generated by exploiting the notion of retraction map introduced in Section 3. A central transference matrix close to group identity may be generated by:

$$\bar{\tau} = R_{e_5}(w), \tag{29}$$

where w is a randomly generated tangent matrix in $T_{e_5}\mathbb{E}$, namely:

$$w = \begin{bmatrix} h & r \\ 0_{4,1}^T & 0 \end{bmatrix},$$

with $h \in \mathbb{H}(4, \mathbb{R})$ and $r \in \mathbb{R}^4$. Hamiltonian matrices may be generated by the rule:

$$h = \frac{1}{2} q(\varepsilon^T + \varepsilon), \tag{30}$$

with $\varepsilon \in \mathbb{R}^{4\times 4}$ being a random matrix with normally distributed entries. A random 'constellation' of matrices τ_n may now be generated via the rule:

$$\tau_n = R_{\bar{\tau}}(w_n), \tag{31}$$

where w_n is a randomly generated tangent matrix in $T_{\bar{\tau}}\mathbb{E}$. Note that any $\bar{\tau}^{-1} w_n$ should be a tangent matrix at identity, thus random tangent matrices w_n may be generated by the rule $\beta \bar{\tau} q(\varepsilon^T + \varepsilon)$, with $\varepsilon \in \mathbb{R}^{4\times 4}$ being again a random matrix with normally distributed entries and $|\beta| < \frac{1}{2}$.

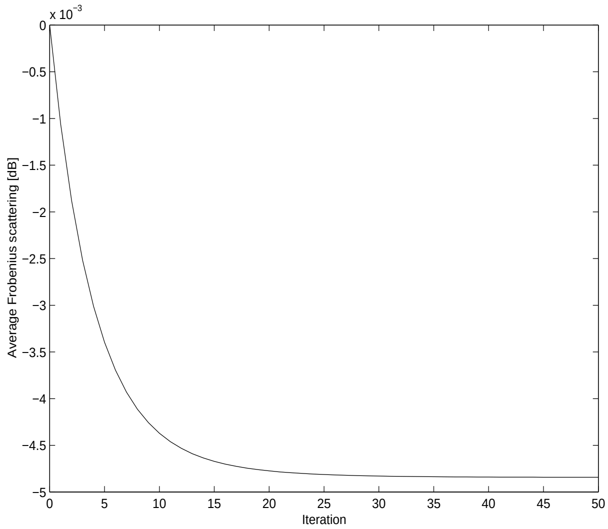

Figure 7: Experiment on averaging random operators over the space $\mathbb{E}$.

Figure 7 displays the result of a numerical experiment obtained with $N = 50$ randomly generated optical transference operators. As is it readily seen from the Figure 7, the averaging algorithm converges steadily and refines Harris' estimate of the empirical ensemble average. The refinement is slight, which suggests that Harris' estimate is nearly optimal already.

4.2 Numerical experiments on averaging model eyes

An experiment described in [5] was repeated by generating a set of optical transference matrices on the basis of Keating-Harris eye model including some randomness. It is interesting to note that in this experiment, the $N = 50$ matrices $\tau_n \in \mathbb{E}$ look quite ill-conditioned.

Figure 8 displays the result of running the algorithm (27). The number of iterations of the algorithm was set to 50. The averaging algorithm converges steadily and refines Harris' average slightly.

5 Conclusions

There is an increasing need of a reliable and precise modeling of optical systems, motivated both by technological and medical applications. The present manuscript aims at describing a solution to the problem of computing the average of a set of optical-system-transference

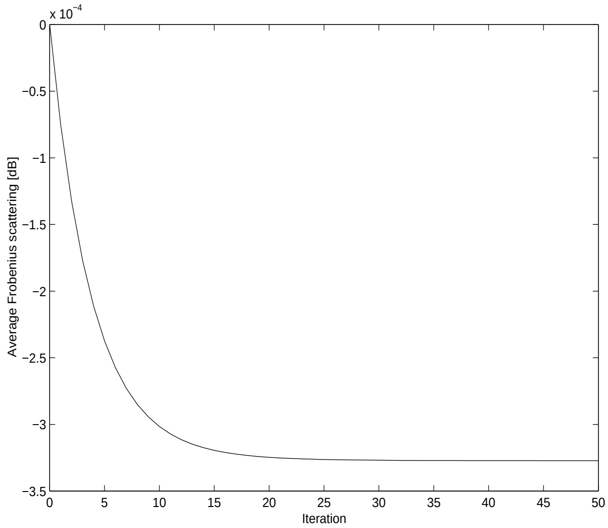

Figure 8: Experiment on averaging model eyes.

matrices. It is shown that such transference matrices form a Lie group that has the structure of a symplectic $\mathbb{S}(4,\mathbb{R})$ group foliated by a Euclidean space $\mathbb{R}^4$.

The present paper builds on a previous non-iterative averaging rule by Harris [8]. Numerical results obtained in the experiments show that Harris' rule is nearly optimal, as iteration by the algorithm (27) can improve Harris' result only slightly.

The present study is based on linear optics and is formulated in terms of general linear optical systems. The main application of the discussed framework is to human eyeball modeling. Although, at present, there is no application of the discussed algorithmic framework to machine vision, it will be worth investigating such possibility in the future.

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