One-Unit 'Rigid-Bodies' Learning Rule for Principal/Independent Component Analysis with Application to ECT-NDE Signal Processing

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Abstract

The aim of this research work is to present a detailed theoretical analysis of the one-unit learning rule based on the rigid-bodies learning theory, specialized for first principal/independent component analysis. The adaptation equations are regarded as generators of weight-flows on a structured parameters space; the stationary points of the learning equations are determined and their stability is proven through a suitable Lyapunov function. The neuron is also excited with both synthetic and real-world signals in order to numerically investigate its behavior, and eddy-current-signal processing is carried out as an application of the developed independent component analysis algorithm to non-destructive evaluation of metallic objects.

Key words: One-unit neural system; Unsupervised learning theory; Independent component analysis; Principal component analysis; Rigid-body dynamics; Pattern recognition; Eddy-current (EC) phenomenon; Non-destructive evaluation (NDE).

1 Introduction

In an early report [17], a new class of learning rules for linear as well as nonlinear neural layers was introduced, which arises from the dynamics of rigid bodies. Their efficiency in solving some orthonormal problems such as optimal data representation by second-order statistics decomposition and blind source separation from non-convolutional mixtures by higher-order statistical

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processing was experimentally proven. Later on, it was observed that the mentioned class of learning algorithms is a subset of a larger family of adaptation rules and a general theoretical framework which explains many related contributions found in the scientific literature was proposed. The general theory, termed *Stiefel manifold and Lie group learning*, was presented in [19,21]. The main idea behind these contributions is to exploit the mathematical knowledge of the geometric structure of the spaces that the networks' parameters belong to, through the basic instruments provided by differential geometry, as recently suggested for instance by Amari [2], Nishimori [31] and Edelman-Arias-Smith [16], among others; our work also found its roots in some impressive papers on second-order optimization techniques (see e.g. [1,35]) exploiting physical parallelisms.

The aim of this paper is to study some particular theoretical aspects of oneunit learning by rigid-bodies (or 'mechanical') rule with reference to the widely investigated topic of first principal/minor component analysis (for an up-todate review see e.g. [10,13,18,32]) and real/complex-valued independent component analysis (see [8,20,23,24,28] and references therein).

As mentioned, the mechanical learning paradigm arises from the equations describing the dynamics of an abstract rigid body, embedded in a force field, which is formed by unitary-mass point-particles positioned over mutually orthogonal axes at unitary distance from axes' origin. If $w(t) \in \mathbb{R}^n$ describes the position at time t of a single mass (or, equivalently, the configuration or internal state of a single neuron), the equations governing system's dynamics write [17]:

$$w' = Aw, \ p = -\mu Aw \ , \ f = -2\nabla_w U \ , \tag{1}$$

$$A' = \frac{1}{4} [(f+p)w^T - w(f+p)^T] .$$
⁽²⁾

In the equations, the prime denotes derivation with respect to time, the superscript T denotes transposition, $A \in \mathbb{R}^{n \times n}$ is a kind of angular speed, $p \in \mathbb{R}^{n}$ represents the braking effect produced by the fluid, permeating the space that the body moves within, having viscosity μ , and $f \in \mathbb{R}^{n}$ represents the force field which makes the body moving. Here it is supposed that the force field derives from a potential energy function $U : \mathbb{R}^{n} \to \mathbb{R}$, which describes the *n*-input/1-output network task.

The basic properties of the system (1)+(2) may be summarized as follows:

• Let us denote by $SO(n, \mathbb{R})$ the special orthogonal group, that is the subset of $\mathbb{R}^{n \times n}$ of the orthogonal matrices with positive unitary determinant; $SO(n, \mathbb{R})$ is a Lie group with Lie algebra $\mathbf{so}(n, \mathbb{R})$, i.e. tangent space at the origin: $\mathbf{so}(n, \mathbb{R})$ is known to be the set of skew-symmetric matrices, that is $\mathbf{so}(n, \mathbb{R}) \stackrel{\text{def}}{=} \{X \in \mathbb{R}^{n \times n} | X^T = -X\}$ [26]. It is now immediate to verify that if $A(0) \in \mathbf{so}(n, \mathbb{R})$ then the equation (2) provides $A'(t) \in \mathbf{so}(n, \mathbb{R})$ and thus $A(t) \in \mathbf{so}(n, \mathbb{R})$, because $\mathbf{so}(n, \mathbb{R})$ is a linear space.

- Let us denote by S^{n-1} the unit-radius sphere, defined as $S^{n-1} \stackrel{\text{def}}{=} \{w \in \mathbb{R}^n | w^T w = 1\}$. Because of the skew-symmetry of A(t) we see from the first of equations (1) that if $w(0) \in S^{n-1}$ then $w(t) \in S^{n-1}$ for all t. Also, if we denote by $T_w S^{n-1}$ the tangent space to the *n*-sphere at w, it readily proves that $p, w' \in T_w S^{n-1}$.
- The equilibrium conditions for the system (1)+(2), i.e. the equilibrium conditions for the learning rule, write $w(t_{\star}) \in \ker\{A(t_{\star})\}$, $f(t_{\star})w^{T}(t_{\star}) w(t_{\star})f^{T}(t_{\star}) = 0_{n}$, $w(t_{\star}) \in S^{n-1}$, $A(t_{\star}) \in \mathbf{so}(n, \mathbb{R})$, where $\ker\{\cdot\}$ denotes the null-space (kernel) of the considered linear operator, the symbol 0_{n} denotes the null element of $\mathbb{R}^{n \times n}$ and t_{\star} denotes the instant in which the equilibrium holds. It is important to recall that both w(t) and A(t) are unknown and that the force-field f(t) is in general a non-linear function of the network's weights, thus the above conditions are the most general equilibrium results.
- As a mechanical system, stimulated by a conservative force field, tends to minimize its potential energy U, the set of learning equations (1)+(2) for a neural unit with weight-vector w may be regarded as a non-conventional optimization algorithm; if C(w) represents the cost associated with neuron's parameters misadjustment with respect to a pre-defined task, assuming $U(w) \propto C(w)$ allows the network to learn the task.

2 The case of first principal/minor component analysis

Let us consider a linear neural unit described by $y(t) = w^T(t)x(t)$, where $x \in \mathbb{R}^n$ denotes the neuron's input random vector and $y \in \mathbb{R}$ describes the neuron's response, at time t. In the hypothesis that the multivariate random vector x is zero-mean and has finite covariance matrix $C_x \stackrel{\text{def}}{=} E[xx^T]$, the first principal component analysis of x consists in the projection of the vector stream over an axis described by w_{pc} such that the variance of the projection is maximal, for a unit-length projector [32,41]:

$$w_{\rm pc} = \arg \max_{w^T w = 1} E_x[(w^T x)^2] .$$

The first minor component analysis searches instead for the direction $w_{\rm mc}$ that the minimum-variance projection corresponds to:

$$w_{\rm mc} = \arg\min_{w^T w = 1} E_x[(w^T x)^2] \; .$$

In this section we recast the above optimization problems as a one-unit rigidbody learning task, determine the stationary solutions of the learning equations and show, through a suitably constructed Lyapunov function, their asymptotic stability within their basins of attraction.

2.1 Potential energy function and force field

In the context of one-unit neural networks, both principal/minor component analysis problems are well-described by the cost function $C(w) \stackrel{\text{def}}{=} \frac{1}{2} \eta E_x[y^2] = \frac{1}{2} \eta w^T C_x w$ (where $\eta > 0$ corresponds to minor component analysis and $\eta < 0$ corresponds to principal component analysis), to be minimized under the constraint that $w \in S^{n-1}$.

In the context of neural learning by rigid-body dynamics, the above optimization problem may be solved by identifying U(w) = C(w), which gives rise to the force field:

$$f = -2\eta C_x w . aga{3}$$

In the following we aim at giving a detailed analysis of this case study. Some interesting remarks are that:

- The stability of the learning system does not depend on the sign of η (contrary to some principal/minor component extraction neural algorithms that do not exhibit such interesting symmetrical behavior [22]).
- Due to their intrinsic properties, second-order (dynamical) systems exhibit a low-pass behavior on input stimuli, which has the same effect of statistical expectation for ergodic signals, thus the same learning system may be employed to solve the stochastic version of the above optimization problem, where C_x is unknown and only x(t) is available at time t, by simply replacing the vector-field f with its stochastic approximation $f^* = -2\eta xy$. The last expression justifies why the vector f^* is referred to as Hebbian forcing term.
- For the sake of completeness, it is worth noting that the extraction of a single component with the proposed learning system is a case study, but the same learning system allows for parallel extraction of several principal/minor components as was illustrated numerically in [17].
- Apart from the sign of the constants η and μ , their values affect the behavior of the learning system. An empirical discussion of their choice may be found in [17], while a formal investigation of their effect on learning dynamics was carried out for some special cases in [21].

2.2 Theoretical analysis: One-unit PCA case

To initiate the analysis, let us rewrite equations (1)+(2) with the force field (3) in a more convenient way. By computing the eigenvalue decomposition of the covariance matrix $C_x = ED^2E^T$, where $E, E^T \in SO(n, \mathbb{R})$ and $D^2 = \text{diag}(D_1^2, D_2^2, \ldots, D_n^2)$ with $|D_1| > |D_2| > \cdots > |D_n|$, it is possible to define the new state variables:

$$v \stackrel{\text{def}}{=} E^T w \ , \ B \stackrel{\text{def}}{=} E^T A E \ . \tag{4}$$

Note that the eigenvalues of the covariance matrix have been supposed distinct, so that the matrix D^2 is invertible. This assumption is motivated by the observation that in most practical signal processing tasks C_x is full-rank. Of course, even if this falls outside the scope of the present contribution, the analysis of the nearly-defective case would be worth considering.

On the basis of these transformations, the learning equations recast into:

$$v' = Bv (5)$$

$$-4B' = 2\eta [D^2, vv^T] + \mu \{B, vv^T\} , \qquad (6)$$

where $v \in S^{n-1}$ and $B, B' \in \mathbf{so}(n, \mathbb{R})$, because the transformations (4) define an isometry on \mathbb{R}^n and a skew-symmetric similarity, respectively; also, we used the commutator and anti-commutator operators defined by $[X, Y] \stackrel{\text{def}}{=} XY - YX$ and $\{X, Y\} \stackrel{\text{def}}{=} XY + YX$, respectively. It is also interesting to note that, by defining $Z \stackrel{\text{def}}{=} vv^T$, the equations above rewrite as:

$$Z' = [B, Z] ,$$

 $-4B' = 2\eta [D^2, Z] + \mu \{B, Z\} ,$

that is in the standard Lie-bracket form [26].

In the new basis, the potential energy function writes $U_E(v) = \frac{1}{2}\eta v^T D^2 v$, while the equilibrium conditions read now:

$$v \in \ker\{B\}, \ [D^2, Z] = 0_n;$$
 (7)

in fact, note that if $v \in \ker\{B\}$ then $\{B, Z\} = 0_n$. As D^2 is diagonal invertible, D^2 and Z commute iff v has the structure $v_{i\star} = \delta_{k,i}, k \in \{1, 2, \ldots, n\}$, where $\delta_{i,k}$ denotes the Kronecker's delta. In this case, the potential energy function simplifies into $U_E(v_\star) \stackrel{\text{def}}{=} \frac{1}{2}\eta D_k^2$. The function $U_E(v)$ minimizes at equilibrium, thus we have two cases: If $\eta > 0$ then k = 1 (i.e. w equals the eigenvector corresponding to the largest neuron's response variance), while if $\eta > 0$ then k = n (hence w corresponds to the minimal neuron-input covariance matrix's eigenvalue).

In order to ascertain the stability of the learning system with the considered force field, let us show how it is possible to construct a Lyapunov function for the system which ensures asymptotic convergence to one of the local minima of U. It is worth recalling that a Lyapunov function is always associated to a basin of attraction of a stationary point, i.e. it ensures convergence provided that the parameters-trajectory originates sufficiently close to the expected solution. In order to show the existence of a positive and decreasing function of the time, note that from relationship (5) it follows:

$$v'' = B'v + Bv' . ag{8}$$

The evaluation of the first term on the right-hand side requires the computation of:

$$2\eta[D^2, Z]v = 2\nabla_v U_E - 4U_E v ,$$

$$\mu\{B, Z\}v = \mu v' ,$$

where the properties $v^T v = 1$ and $v^T B v = 0$ were used. Let us now evaluate the term $dv^T v''$: On the basis of the above equivalence we have $-4dv^T B' v = 2U'_E dt + \mu ||v'||_2^2 dt$, while $dv^T B v' = 0$. As a consequence, the following identity holds true:

$$-4(v')^T v'' dt = 2U'_E dt + \mu ||v'||_2^2 dt .$$

By integrating both sides of the above equation with respect to the time, over the interval [0, t], the following balance equation is readily obtained:

$$-2\|v'\|_2^2|_0^t = 2U_E|_0^t + \mu \int_0^t \|v'\|_2^2 dt .$$

Hence, by the definition of neuron's kinetic energy $K(t) \stackrel{\text{def}}{=} \frac{1}{2} ||v'(t)||_2^2 = -\frac{1}{2}v^T(t)B^2(t)v(t)$, we find the final relationship:

$$K(t) - K(0) = -\frac{1}{2} [U_E(t) - U_E(0)] - \frac{1}{2} \mu \int_0^t K(\tau) d\tau .$$
(9)

A Lyapunov function H(t) for the neuron writes thus:

$$H(t) \stackrel{\text{def}}{=} K(t) + \frac{1}{2} [U_E(t) - U_E^{\min}] , \qquad (10)$$

where U_E^{\min} is the minimum value of $U_E(t)$ within the basin of attraction; note that the existence of the minimum is certain because U_E is a continuous function defined on a compact manifold (the hyper-sphere S^{n-1}). The Lyapunov function is nothing but the Hamiltonian of the neuron: It writes as the sum of two non-negative continuous quantities, thus $H(t) \ge 0$, moreover, it satisfies $H(t) = H(0) - \frac{1}{2}\mu \int_0^t K(\tau) d\tau$, thus it also holds:

$$H'(t) = -\frac{1}{2}\mu K(t) \le 0 , \qquad (11)$$

where the derivative zeros iff $v \in \ker\{B\}$; also, with reference to each basin of attraction for v, H(t) minimizes at the local minima of U_E , as anticipated.

2.3 Extension to robust principal/minor component analysis

An useful extension of the above-developed theory, concerning the definition of a potential energy function that still makes the single-unit network able to compute principal/minor-like components, obtains by replacing the optimization of the variance of neuron's response with the optimization of an average non-quadratic function of neuron's response. This allows for *robust* principal/minor components estimation, that helps alleviating the estimation difficulties related to noise, disturbances, and statistical outliers [29,33]. Neuron's learning may be performed with equations (1)+(2) and its behavior depends on the shape of the cost functions to be optimized corresponding to the selected potential energy function.

As an example, let us consider the case that the search for the minima of $C_{\rm R}(w) \stackrel{\text{def}}{=} \eta E_x[\Psi(w^T x)]$ drives neuron's learning, where $\Psi(\cdot)$ is a monotonically increasing non-negative continuous function, with a only minimum in 0, increasing no more than quadratically, i.e. $\Psi(y) \leq y^2/2$ [29]. In this case, if $p_x(x)$ denotes the probability density distribution of the multivariate neuron's input, we have:

$$0 \le \frac{2}{\eta} C_{\mathbf{R}}(w) = 2 \int_{\mathbf{R}^n} \Psi(w^T x) p_x(x) dx^n \le \int_{\mathbf{R}^n} (w^T x)^2 p_x(x) dx^n = \frac{2}{\eta} U(w) ,$$

thus the quadratic potential energy U(w) makes the Hamiltonian (10) be again

a valid Lyapunov function to prove the existence of asymptotically stable neural configurations.

2.4 An illustrative computer simulation

As an illustrative example, let us consider the results obtained with:

$$C_x = \begin{pmatrix} 0.9 \ 0.4 \ 0.7 \\ 0.4 \ 0.3 \ 0.5 \\ 0.7 \ 0.5 \ 1.0 \end{pmatrix}, \ w(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ A(0) = 0_3 ,$$

which is a sub-case of the example proposed by Chen, Amari and Lin in [10].

In the first experiment we chose $\eta = -0.5$ and $\mu = 4$ in order to extract the principal eigenvector. The learning equations have been discretized in time with $\Delta t = 0.001$. The results shown in the first row of Figure 1 illustrate the behavior of functions U_E and H, as well as of the state-vector w(t), which are as expected. In fact, the Lyapunov function H(t) vanishes to zero when the algorithm reaches its steady-state, thus K(t) vanishes when the potential energy function $U_E(t)$ minimizes. The stochastic version of the same learning rule, endowed with the Hebbian forcing term, has been implemented, too: The obtained numerical results are coherent.

The second row of Figure 1 refers instead to the selection $\eta = 3$, $\mu = 6$, $\Delta t = 0.0015$, in order to extract the minor eigenvector. Again the results are as expected, both for the non-stochastic and stochastic cases.

3 The case of independent component analysis

The independent component analysis (ICA) aims at extracting independent signals from their linear mixtures or to extract independent features (as latent variables) from signals having complex structure. Published books on ICA (*in quo totum continetur*) are [8,28]. A recent overview of independent component analysis, along with a discussion of its relationship with *factor analysis* and *projection pursuit*, has been recently presented in [23].

A way to design an independent component analysis method is to employ the maximum or minimum kurtosis principle: Under some conditions [12], the output of a linear neuron with multiple inputs x(t) described by y(t) =



Fig. 1. Simulation results for first principal component analysis (fpca) and first minor component analysis (fmca). The spheres are used to visualize the pathways followed by neuron-state w during the learning phase. The graphs on the left show the values of the potential energy function U_E (dot-dashed line) and the Lyapunov function H (solid line) during learning.

 $w^{T}(t)x(t)$ contains an independent component of the input if the weight-vector w maximizes or minimizes the fourth moment of neuron response:

$$w_{\rm ic} = \arg \max_{w^T w = 1} \pm E_x[(w^T x)^4]$$
 (12)

Such technique derives from the knowledge of the properties of statistical cumulants and their transformation through a linear MIMO (multiple-input, multiple-output) system (see e.g. [14]).

The observed signal model is x(t) = Ms(t), where $s(t) \in \mathbb{R}^n$ is a vectorsignal with statistically independent components, and $M \in \mathbb{R}^{m \times n}$ is a fullcolumn rank matrix describing the mixing of the *n* independent components into the *m* observable signals, or the expected relationship between the latent variables and the observable variables. Apart from special cases (namely in under-determined ICA, where additional assumptions are necessary for source recovery) the number of observations *m* should exceed or equate the number of independent sources *n*. With the convention that $s_r(t)$ denotes the r^{th} component of the vector s(t), usually the hypotheses made on the source stream are that each s_r is an ergodic stationary IID (independent, identically distributed) random signal with zero mean $(E_s[s_r] = 0)$, unitary variance $(E_s[s_r^2] = 1)$ and is statistically independent of each other at any time. It is also worth recalling that, under the above hypotheses, the *kurtosis* of the signal s_r defines as $\kappa_4^r \stackrel{\text{def}}{=} E_s[s_r^4] - 3$. The value of the kurtosis gives rise to a classification: When a signal has negative kurtosis it is termed sub-Gaussian or *plati-kurtotic*, while a signal having positive kurtosis is termed super-Gaussian or *lepto-kurtotic*; a signal with zero-kurtosis is termed *meso-kurtotic*: Gaussian signals are mesokurtotic, but of course a signal needs not to be Gaussian to have zero-kurtosis.

In practical situations, it is also common to hypothesize that the signals in x are mutually uncorrelated, which is equivalent to say that the mixing matrix is orthogonal; without any loss of generality we can also suppose m = n, thus ultimately $M \in SO(n, \mathbb{R})$. It is worth mentioning that when the observable signals are not uncorrelated, a pre-processing stage known as 'whitening' or 'sphering' may be always performed, which has the twofold effect to remove the second-order dependency between the signals and to reduce the number of 'geometrically independent' observations to n.

3.1 Potential energy function and force field

In this section, we again recast the original optimization problem into a oneunit rigid-body learning task, determine the stationary solutions of the learning equations and show their asymptotic stability within their basins of attraction.

In the present context, the optimization principle (12) gives rise to the potential energy function:

$$U(w) \stackrel{\text{def}}{=} \frac{1}{4} \eta \{ E_x[(w^T x)^4] - 3 \} , \qquad (13)$$

with η being again a real number allowing to switch between the maximization and minimization problems. The above energy function generates the following forcing field along with its stochastic approximation:

$$f = -2\eta E_x[(w^T x)^3 x], \ f^* = -2\eta x y^3.$$
 (14)

As in the previous section, in the following we wish to investigate the structure and the behavior of mechanics-like learning equations under the above force field. In the present context it is worth performing the variable changes $v \stackrel{\text{def}}{=} M^T w$ and $B \stackrel{\text{def}}{=} M^T A M$, which allows recasting the fundamental learning equations (1)-(2) into:

$$v' = Bv \tag{15}$$

$$4B' = -2\eta E_s[y^3(sv^T - vs^T)] + \mu\{B, vv^T\} , \qquad (16)$$

while the potential energy function, in the new basis, writes:

$$U_M(v) \stackrel{\text{def}}{=} \frac{1}{4} \eta \{ E_s[(v^T s)^4] - 3 \} .$$
 (17)

Note that, by the hypotheses, M is an orthonormal matrix, thus the transformation of w into v describes an isometry in \mathbb{R}^n , hence $w \in S^{n-1}$ implies $v \in S^{n-1}$, thus v(t) belongs to the unit *n*-sphere at any time. Also, the fact that the equations may be rewritten equivalently in term of s means that the behavior of the learning system is independent of the mixing operator M, that is an important property known as *equivariancy* [9] (in other terms, it may be envisaged that the independent component analysis algorithm under investigation provides an equivariant estimation of the source stream).

From equations (15)-(16) it follows that the equilibrium conditions for the learning system are:

$$v \in \ker\{B\}$$
, $E_s[y^3s]v^T - vE_s[y^3s^T] = 0_n$, (18)

where $y = v^T s$. The second condition would imply the symmetry of the matrix $E_s[y^3s]v^T$. In order to write this constraint in a more expressive way, it pays to compute each of the factors $E_s[y^3s_r]$ in closed form. Denoting with v_r the entries of vector v, on the basis of the hypotheses made on the multivariate signal s, we have:

$$\begin{split} E_s[y^3s_r] &= \sum_i \sum_j \sum_k v_i v_j v_k E_s[s_i s_j s_k s_r] = \\ &= E_s[s_r^4] v_r^3 + 3(1 - v_r^2) v_r = \kappa_4^r v_r^3 + 3v_r \end{split}$$

The $(i, j)^{\text{th}}$ component of $E_s[y^3s]v^T$ has thus the expression $(\kappa_4^i v_i^3 + 3v_i)v_j$, therefore the second equilibrium condition in (18) rewrites:

$$(\kappa_4^i v_i^2 - \kappa_4^j v_j^2)(v_i v_j) = 0 , \ \forall (i,j) \in \{1, 2, ..., n\}^2 .$$
(19)

The above conditions are clearly fulfilled for those values of the indexes i and j such that $v_i = 0$ (so that the second parentheses vanish) or when i = j (so that the first parentheses vanish), thus they define the subset $\mathcal{N} \stackrel{\text{def}}{=} \{i \in \{1, ..., n\} | v_i \neq 0\}$. This, in turn, induces the definition of a bijective map $R(r) : \mathcal{N} \to \{1, ..., n\}$ that assigns to each index r of a non-null element of v one, and only one, index of an element in the same vector that is in correspondence with it on the basis of the relationship $\kappa_4^r v_r^2 - \kappa_4^{R(r)} v_R^2(r) = 0$. Note that such a map is not unique.

A point v_{\star} in \mathbb{R}^n that of course satisfies conditions (19) is any canonical vector of \mathbb{R}^n , that is $v_i = \pm \delta_{\bar{r},i}$, that gives $U_M(v_{\star}) = \frac{\eta}{4} \kappa_4^{\bar{r}}$: As the system seeks for the minimum of U_M , the value \bar{r} corresponds to the most lepto-kurtotic independent signal if $\eta < 0$, or the most plati-kurtotic independent signal if $\eta > 0$. As a matter of fact, as $y = v^T s$, the result $v_i = \pm \delta_{\bar{r},i}$ implies that $y_i = \pm s_{\bar{r}}$, therefore the neuron is able to separate out one of the independent signals that presented mixed to other signals at its input.

This equilibrium point is the only extremal one, in fact the other vectors satisfying (19) are sub-optimal. In order to show this property of the potential energy function, it pays to express the potential function in closed form by taking into account that:

$$E_{s}[(v^{T}s)^{4}] = \sum_{i} \sum_{j} \sum_{k} \sum_{r} v_{i}v_{j}v_{k}v_{r}E_{s}[s_{i}s_{j}s_{k}s_{r}] =$$
$$= \sum_{r} E_{s}[s_{r}^{4}]v_{r}^{4} + 3\sum_{r}(1-v_{r}^{2})v_{r}^{2},$$

thus ultimately $U_M(v) = \frac{1}{4}\eta \sum_r \kappa_4^r v_r^4$; note that the summation may run over $\{1, ..., n\}$ or \mathcal{N} , as well. This expression allows us to show the sub-optimality of the equilibria different from the canonical vectors: In fact, the potential energy function computed in a point v_{\star}^{sub} satisfying (19) readily proves to write:

$$U_M(v_\star^{\rm sub}) = \frac{\eta}{4} \sum_{r \in \mathcal{N}} \kappa_4^{R(r)} v_r^2 v_{R(r)}^2 .$$
 (20)

The fact that $\sum_{r} v_{r}^{2} = 1$ implies $v_{r}^{2} \leq 1$, thus the majorization $4|U_{M}(v_{\star}^{\mathrm{sub}})/\eta| \leq \sum_{r \in \mathcal{N}} |\kappa_{4}^{R(r)}| v_{R(r)}^{2}$ holds true; also, the equality may hold only when $v_{R(r)} = \pm \delta_{r,\hat{r}}$ for some \hat{r} . By denoting $\bar{r} = \arg \max_{r \in \mathcal{N}} \{|\kappa_{4}^{R(r)}|\}$, we arrive at the conclusion that $4|U_{M}(v_{\star}^{\mathrm{sub}})/\eta| \leq |\kappa_{4}^{R(\bar{r})}|$, when the equality may hold only if $v_{R(r)} = \pm \delta_{r,\bar{r}}$

Equations (18) characterize the equilibrium points of the learning system, and we wonder if it is possible to prove the existence of a Lyapunov function for the system ensuring its convergence (at least locally). The Hamiltonian of the neuron is again a good candidate, provided that it satisfies the balance equation (9). Although this equation was demonstrated for the specific case of a quadratic potential energy function, it may be proven that it holds irrespective of the structure of the potential energy field [19,21]. It follows that the form (10) does represent a valid Lyapunov function for the system, in fact, the function $U_M(v)$ is continuous and defined on a compact manifold and thus possesses a minimal value, therefore H(t) may be defined to be positive and again it is such that $H'(t) \leq 0$, ensuring the convergence of the learning system in the one-unit ICA case, too.

3.3 Illustrative numerical results on one-unit ICA

As a case-study, let us consider the case that n = 2 and that the source signals have kurtoses $\kappa_4^1 = 1$ and $\kappa_4^2 = 5$, and let us assume $\eta = 4$. In this case the rotated-basis weight-vector has components $v = [v_1 \ v_2]^T$, which satisfy $v_1^2 + v_2^2 = 1$. Due to the sign-blindness of the ICA, we can restrict our analysis to $v_r \in [0, 1]$. If v differs from a canonical vector, both v_1 and v_2 differ from zero, thus the equilibrium condition would imply $v_1^2 = 5v_2^2$: The only point on the circle S^1 satisfying this constraint is $v_\star = \left[\sqrt{\frac{5}{6}} \sqrt{\frac{1}{6}}\right]^T$, where the potential energy function equals $-\frac{26}{36}$ that is greater than -5. This means that the solution v_\star is sub-optimal and, therefore, the only optimal solutions remain the canonical vectors.

Another interesting example is the following: Let us suppose, again in the 2sources problem, that $\kappa_4^1 = \kappa_4^2 = \bar{\kappa} > 0$ and $\eta = 4$. In this case the equilibria are at $v_1^2 = v_2^2$ thus the only non-canonical feasible point in the positive octant is $v_{\star} = \left[\sqrt{\frac{1}{2}} \sqrt{\frac{1}{2}}\right]^T$ (irrespective of the value of $\bar{\kappa}$); in this point the potential energy function has the value $-\bar{\kappa}/2 > -\bar{\kappa}$, thus, again v_{\star} is a sub-optimal configuration.

Let us now show two computer-based simulations that should serve as illustrations for the two cases-study just discussed. The first one concerns the implementation of learning system with forces expressed in closed form, which allows controlling exactly the sign and the magnitude of the source-signals' kurtoses (in the 2-sources problem). It is worth to observe that the forcing term writes symbolically $f = -2\eta(\kappa_4^s \circ v^3 + 3v)$, where κ_4^s denotes the kurtoses vector, namely $\kappa_4^s \stackrel{\text{def}}{=} [\kappa_4^1 \kappa_4^2]^T$, the $(\cdot)^3$ -exponentiation acts component-wise, and the symbol \circ denotes the Hadamard product ¹. Now, the learning equation

 $[\]overline{}^{1}$ The Hadamard product between two tensors of equal size is a tensor of the same size obtained by component-wise multiplication of the two factors.

(16) involves the term $fv^T - vf^T$, which readily proves to simplify into:

$$fv^T - vf^T = -2\eta[(\kappa_4^s \circ v^3)v^T - v(\kappa_4^s \circ v^3)^T]$$

The result of the numerical simulations are shown in the Figure 2, in the two different situations just discussed. The learning parameters have values $\mu = 4$



Fig. 2. Simulation results for one-unit independent component analysis: Controlled experiments over synthetic signals. Left: Values of the potential energy function U_E (dot-dashed line) and the Lyapunov function H (solid line); Right: Course of the rotated-basis weights v_1 (solid) and v_2 (dashed).

and $\eta = -0.4$, and the sampling-step has value $\Delta t = 0.001$.

Another interesting experiment concerns the extraction of an independent signal from a mixture of real-world signals, which are four gray-level 100×100 digital images. The original images in the multivariate signal $s \in \mathbb{R}^4$ are shown in the Figure 3, and their kurtoses are (from left to right in the Figure): 1.9809, -0.7761, -0.8430 and -1.6713.

It is well-known that the separation of real-world natural images is often a difficult task, because they represent neither IID sequences nor completely independent source signals. To show this in the present case, we computed for instance the covariance matrix of the four vectorized images: By taking into account that the pixel-values range in [0, 255], it is in fact worth examining



Fig. 3. Original natural images (top) used in the four-source-separation problem, and a linear mixture of them (bottom).

the structure of the covariance $C_s \stackrel{\text{def}}{=} E_s[ss^T]$, that has been computed to be:

$$C_s = 10^3 \times \begin{bmatrix} 3.1878 & 1.0898 & 0.0416 & -0.5739 \\ 1.0898 & 2.6651 & 0.0148 & -0.6820 \\ 0.0416 & 0.0148 & 2.1062 & 0.0036 \\ -0.5739 & -0.6820 & 0.0036 & 4.4301 \end{bmatrix}$$

;

it looks non-diagonal, confirming that any source is highly correlated to each other. Also, in the Figure 4 it is shown the auto-correlation function of the first image reported in Figure 3: Its non-sharp shape confirms its non-IID nature.

By choosing $\eta > 0$ in the algorithm we can extract the least kurtotic signal, while choosing $\eta < 0$ allows extracting the most kurtotic image from the mixture. In order to quantitatively assess the result of this experiment, the interference-to-signal ratio (ISR) was used as performance measure, which is defined as:

$$ISR \stackrel{\text{def}}{=} \frac{\|w^T M\|_2^2}{\|w^T M\|_{\infty}^2} - 1 \ge 0 \ .$$

It bases on the knowledge that the optimal (separating) weight-vector w must cancel the contribution given by matrix M except that for one column; in the above formula, notation $\|\cdot\|_{\infty}$ denotes the L_{∞} (max-abs) norm. The



Fig. 4. Auto-correlation function of the signal obtained by vectorizing the first of source images in Figure 3 and by previous mean-value removal.

interference-to-signal ratio is non-negative and its nullity holds only when one component has been perfectly extracted from the mixture². In a realworld context, of course some residual interference should be tolerated. Also, during the neuron's learning phase the value of neuron's kinetic energy K(t) is recorded, as it represents the internal state of the neuron, which should quiet after sufficiently long time as the learning progress goes on.

The Figure 5 refers to the least-kurtotic image extraction ($\eta = 0.4$, $\mu = 4$, $\Delta t = 0.001$) and to the most-kurtotic image extraction ($\eta = -0.5$, $\mu = 20$, $\Delta t = 0.001$) from the four-images linear mixture. These results have been obtained with a training-set containing 10,000 input patterns randomly picked during the training in order to emulate stationarity [4], which helps to alleviate the problem of time-correlated observations³. Figure 6 depicts the two extracted images, which have been recovered faithfully from the linear mixture.

3.4 Extension to multiple-component extraction and non-stationary case

The above sections described the procedure for extracting one component on the basis of the maximum/minimum kurtosis discrimination principle. An useful extension should allow for the estimation of the desired number of in-

² In fact, note that $ISR = ||v||_2^2/||v||_\infty^2 - 1$. For a canonical vector $v = \delta_{i,r}$, therefore, it holds ISR = 0.

³ It is worth noting that, when dealing with instantaneous (i.e. non-convolutional) mixtures, this practical 'trick' may be always exploited in order to improve the learning convergence speed because it does not destroy the structure of the data. After learning, for source recovery, of course the neuron must be presented with the data in the right (natural) order.



Fig. 5. Least-kurtotic and most-kurtotic image extraction: Neuron kinetic energy K (left-hand column) and ISR index (right-hand column) versus time.



Fig. 6. Least-kurtotic and most-kurtotic image extraction: Recovered images.

dependent components. However, this aspect opens the question of the exact knowledge of the number of independent signals which give rise to the mixture, and of the possible time-variability of this number (in a non-stationary environment like in a telecommunication scenario, sources may continually appear and disappear). It is not the aim of this paper to discuss in detail these topics, but we wish to only cite two interesting solutions which have a general span and may be applied independently of the kind of employed one-unit extraction engines.

A solution to the multiple-component extraction based on a one-unit neural engine exploits the property of linearity of the observed-signal model, and is based on the well-known mechanism of *deflation*. It allows the sequential extraction of one component at a time (see e.g. [38] and references therein); in our experience with this method, it suffers from the progressive accumulation of numerical errors and estimation inaccuracies from component to component, thus it is reliable only in presence of a limited amount of components (see e.g. [24]).

As mentioned, the problem of the estimation of the number of sources in a mixture can be solved by second-order statistics analysis: An advantage of signal pre-whitening by the eigenvalue decomposition of the observed-signal covariance matrix is that a careful analysis of the spectrum of the covariance may provide a very accurate estimate of the number of sources, either in the case of noiseless or moderately noisy mixtures. A pervasive example of application of such technique may be found, for instance, in [34], where a non-stationary blind source recovery case was tackled by tracking over time the number of independent components.

4 Application to non-destructive evaluation (NDE) problem

The eddy-current testing (ECT) [5,6] is a non-destructive evaluation (NDE) technique especially well suited for metallic object inspection by a probe system. The probe usually consists of a source coil and a pick-up coil connected to a voltmeter. The probe allows for complex-voltage measurements whose change is used for defect detection and identification with particular interest into defect shape.

In ECT-NDE, the probe is slid over a conductive object. The exciter coil is driven with medium-range frequency (< 100kHz) sinusoidal current producing a magnetic field that induces eddy currents within the object near the exciter. These currents produce their own magnetic fields, which are always in opposition to the exciter field. A part of the eddy currents experiences conductive losses, therefore these counter-fields do not fully balance the exciting field. This phenomenon may be equivalently thought of as the interrogating magnetic field which is back-scattered by the inner layers of the objects [11,39]. At the level of the coil, the back-scattering phenomenon results in an impedance change, which is composed by an equivalent differential resistance, accounting for the energy loss in the metal, and a differential reactance that accounts for the phase delay in the scattered field. The differential impedance is sensitive to anomalies or perturbations (flaws) in the volume along the path of the interrogating magnetic field, such as metal loss, cracks, corrosion or thinning.

A typical inspection is carried out in the following way. A conductive specimen is supposed to be affected by a hidden defect located deeply in its volume: A probe is moved on a grid over an accessible surface of the specimen and a set of differential complex voltage values are thus collected. A strong discontinuity in the homogeneity of the impedance profile in a spatial location clearly evidences the presence of a defect in that zone of the volume. On the basis of this observation, a first automatic screening of the data is performed in order to roughly localize an area of the surface centered around the defect, so that the successive finer analysis is restricted to a narrower specimen's volume. As the distribution of the impedance depends on the location and shape of the defect, it is possible to reconstruct the flaw's profile by properly treating the measured data [27].

When a defect is present on the surface of the specimen, in order to prevent the evolution of the damage, it is important to detect, localize and size the crack. However, the eddy current measurement might be corrupted by the skin effect, the lift-off noise and by the generic uncorrelated noise. Prior to develop a flaw detection/recognition system, each measure has thus to be restored, by separately featuring e.g. the lift-off signal and the defect signal.

In the present work, the magnitude and the phase of the ECT signals, acquired on the upper and lower sides of a specimen, have been considered as available measures. In order to extract information from the measured data, a proper signal processing algorithm should be designed. An ECT-NDE data processing approach is proposed in this paper to remove the effects of the eddy-current sensor drift during the horizontal/vertical scanning of an inspected metallic plate.

Artificial neural networks based techniques have recently been applied to the solution of electromagnetic problems (see e.g. [3,7,37,39] and references therein), and it has been especially proven, by recent experimental research works, that the use of ICA enables us to acquire additional knowledge from measurements [25,36,40]. For further reading, a recent survey of successful industrial applications of independent component analysis and blind source separation may be found in [15].

4.1 Experimental set-up for the NDE problem

We analyze a set of experimental ECT-NDE data, provided by the Hungarian Academy of Sciences [30]. The data have been acquired by a single pancake exciting coil with FLUXSET sensor (for a detailed explanation of experimental set-up see [30]). The tested specimen consists of a square plate ($8 \times 8 \times 0.125$ cm) of INCONEL material, which presents a rectangular thin crack (about 0.2 mm thick and 9 mm in length), located in a region of 2×2 mm width around the plate center. The depth of the defect is about 20% of the plate thickness. The scanned area is a region of 40×40 mm with 0.5 mm spacing

along x and y axes; the output complex-voltage has been recorded on a grid of 81×81 measurement points.

Figures 7 and 8 represent the magnitude abs(V) and the phase arg(V) of the ECT-NDE voltage-signal in the cases of inner and outer defect, respectively. Even if the plate has a constant horizontal thickness, the signal has a magnitude related not only to the defect but also to the sensor lift-off that varies over the specimen surface: This creates a drift effect on the measurements.



Fig. 7. Magnitude and the phase of the ECT-NDE signal in the cases of inner defect (ID). Two different views. (Top: Three-dimensional differential voltage surface. Bottom: Projection over the y-z axis.)

From the experimental set-up, four different measurements are available: The magnitude and the phase of the EC signal acquired from the inner side of plate, and the magnitude and the phase of the signal pertaining to the outer side of the plate. If a defect is present near one of the surfaces, a measure can be considered as ID (inner defect) and the second one will be, necessarily, of OD (outer defect) type. By using a single measurement, the detection of the crack is not allowed because of two concurring problems:

• When the defect is located near the surface on the same side of the sensor (ID), although the signal-to-noise ratio is high, it does not suffice to provide the detection/recognition system a suitable knowledge to correctly locate and size the flaw; indeed, as illustrated in the Figures 7 and 8, the magnitude



Fig. 8. Magnitude and the phase of the ECT-NDE signal in the cases of outer defect (OD). Two different views. (Top: Three-dimensional differential voltage surface. Bottom: Projection over the y-z axis.)

and phase of the ID signal are corrupted by the lift-off noise, and the signal takes its maximum value when the probe is close to the surface, rather than in correspondence of the defect.

• When the defect is located far from the surface, that is for OD measurement, the signal related to the crack is completely buried into background disturbance, due to the skin effect, to the lift-off noise and to the uncorrelated Gaussian noise, as can be readily seen in the Figure 8.

If the electronic devices used to acquire the measures stay the same during the ID/OD scans, we can suppose that, as opposite to random Gaussian noise, the lift-off noise affects both the measurements in the same way. Thus, we can make the reasonable hypothesis that the measured signals are linear mixtures of different sources: The signal related to the defect and the one related to the lift-off noise [36]. Of course, the involved signals as well as the mixing proportions are unknown. This suggests that, on the basis of the ICA technique, a way can be envisaged to extract the defect signal. As mentioned, we hypothesize a linear model relating the independent signals with the measures. Our proposal for processing the available data is to suppose that the real and imaginary parts of the involved signals interact in an additive way, thus the input signal $x(t) \in \mathbb{R}^4$ to the neural network has the structure:

$$x \stackrel{\text{def}}{=} \begin{bmatrix} \operatorname{abs}(V)_{\mathrm{ID}} \cos[\operatorname{arg}(V)_{\mathrm{ID}}] \\ \operatorname{abs}(V)_{\mathrm{OD}} \cos[\operatorname{arg}(V)_{\mathrm{OD}}] \\ \operatorname{abs}(V)_{\mathrm{ID}} \sin[\operatorname{arg}(V)_{\mathrm{ID}}] \\ \operatorname{abs}(V)_{\mathrm{OD}} \sin[\operatorname{arg}(V)_{\mathrm{OD}}] \end{bmatrix}$$

As shown in the previous sections, the first step consists in evaluating first- and second-order statistics. Numerically we obtain $E_x[x] = [-0.0224 - 0.0246 - 0.0611 - 0.0711]^T$ and the estimated covariance matrix:

$$C_x = 10^3 \times \begin{bmatrix} 0.3587 \ 0.3190 \ 0.1577 \ 0.1524 \\ 0.3190 \ 0.2850 \ 0.1405 \ 0.1365 \\ 0.1577 \ 0.1405 \ 0.0821 \ 0.0770 \\ 0.1524 \ 0.1365 \ 0.0770 \ 0.0754 \end{bmatrix}$$

The signal x has thus been first whitened by mean-value removal and by the eigenvalue-decomposition-based normalization of the covariance matrix, so that the whitened data have unitary covariance. Also, it is worth analyzing the temporal correlation of the observed samples: As an example, Figure 9 depicts the auto-correlation function of the signal $x_3(t)$. As the signal shows a non-negligible temporal correlation, the one-unit ICA algorithm is run over 15,000 samples randomly picked from the set of $81 \times 81 = 6561$ available measures, again to simulate stationarity.

The algorithm was run with the parameter-values $\mu = 5$, $\Delta t = 0.001$, $\eta = -0.5$; also, the entries of $w(0) \sim \mathcal{N}(0, 1)$ (and normalized to have unit norm), while $A(0) = 0_4$.

Figure 10 shows the result of a single run: The obtained latent component clearly pertains to the defect signal, which appears no longer buried by lift-off noise. The linear superposition of the measured signals which cancels the background disturbance is non-trivial, as the final neuron weight-vector resulted to be $w = [-0.2093 - 0.0326 \ 0.9094 - 0.3583]^T$.



Fig. 9. Auto-correlation function of signal x_3 in the ECT-NDE problem (imaginary part of ID measure) after mean-value removal.



Fig. 10. Estimated latent variable in the NDE problem corresponding to the defect signal (five different views). The picture on the upper-left corner depicts the value of neuron kinetic energy during the learning phase.

After such ICA-based pre-processing of the available data, the signal-to-noise (i.e. defect signal to background noise) ratio is good enough to enable an automatic recognition system to locate and describe the crack, as shown for instance by the section at x = 0 mm of Figure 10.

5 Conclusion

A special case of rigid-bodies learning theory was analyzed, which allows a single neuron to perform the first-principal/independent component analysis of its inputs. The learning equations, arisen from the study of the dynamics of a mechanical system, have been proven to generate convergent flows on the unit hyper-sphere.

Computer-based experiments have been illustrated and discussed in order to show their numerical behavior, with particular emphasis to image processing and ECT-data processing for non-destructive evaluation. Other potentially interesting applications of the independent analysis technique to electromagnetic data processing problems have recently been proposed, such as the enhancement of the synthetic aperture radar (SAR) imagery [23] and the electromagnetic pollution source analysis for environmental electromagnetic compatibility monitoring purposes [25].

Further promising electrical-engineering applications of one-unit independent component analysis, to unsupervised pattern classification and image de-blurring, are currently under investigation.

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